Lineære differentialligninger af 1. orden

I Steens noter har differentialligningen formen:
$y' = g(x) \cdot y + p(x)$

I bog A har differentialligningen formen:
$y' + a(x) \cdot y = b(x)$

NB: lineær i y, ikke nødvendigvis i x.

Generel løsning (kaldet "panserformlen")

$y' = g(x) \cdot y + p(x)$

```
> restart
> dsolve(y'=g(x)·y+p(x))
```

$$y(x) = \left(\int p(x)\, e^{-\left(\int g(x)\, dx\right)}\, dx + _C1 \right) e^{\int g(x)\, dx} \qquad \textbf{(1.1.1)}$$

Dvs. **løsningen** er: $y(x) = \left(\int p(x)\, e^{-\int g(x)\, dx}\, dx + c \right) e^{\int g(x)\, dx}$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

Løsningen kan også skrives: $y(x) = \left(\int p(x)\, e^{-G(x)}\, dx + c \right) e^{G(x)}$, hvor $G(x)$ er en stamfunktion til $g(x)$, dvs. $G(x) = \int g(x)\, dx$

$y' + a(x) \cdot y = b(x)$

```
> restart
> dsolve(y'+a(x)·y=b(x))
```

$$y(x) = \left(\int b(x)\, e^{\int a(x)\, dx}\, dx + _C1 \right) e^{\int (-a(x))\, dx} \qquad \textbf{(1.2.1)}$$

Dvs. **løsningen** er: $y(x) = \left(\int b(x)\, e^{\int a(x)\, dx}\, dx + c \right) e^{-\int a(x)\, dx}$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

Løsningen kan også skrives: $y(x) = \left(\int b(x)\, e^{A(x)}\, dx + c \right) e^{-A(x)}$, hvor $A(x)$ er en stamfunktion til $a(x)$, dvs. $A(x) = \int a(x)\, dx$

Eksempel 49 side 50 i bog A

Differentialligning: $\frac{dN}{dt} = (0.025 - 0.0004 \cdot t) \cdot N(t)$, hvor $N(0) = 113 \cdot 10^6$

```
> restart
> DiffLign := d/dt N(t) = (0.025 - 0.0004*t)*N(t)
```

$$DiffLign := \frac{d}{dt} N(t) = (0.025 - 0.0004\,t)\,N(t) \quad \textbf{(2.1)}$$

```
> Betingelse := N(0) = 113*10^6
```

$$Betingelse := N(0) = 113000000 \quad \textbf{(2.2)}$$

```
> dsolve({DiffLign, Betingelse}); evalf(%)
```

$$N(t) = 113000000\ e^{-\frac{1}{5000}\,t\,(-125+t)}$$

$$N(t) = 1.13000000\ 10^8\ e^{-0.0002000000000\,t\,(-125.+t)} \quad \textbf{(2.3)}$$

```
> expand(-0.0002000000000*t*(-125.+t))
```

$$0.02500000000\,t - 0.0002000000000\,t^2 \quad \textbf{(2.4)}$$

Konklusion: $N(t) \approx 1.13 \cdot 10^8\ e^{0.0250 \cdot t - 0.000200 \cdot t^2}$

```
> subs(t=30, rhs((2.3))); evalf(%)
```

$$1.13000000\ 10^8\ e^{0.5700000000}$$

$$1.998141768\ 10^8 \quad \textbf{(2.5)}$$

Konklusion: $N(30) \approx 2.00 \cdot 10^8$

Eksempel 50 side 51 i bog A

Differentialligning: $y' = \frac{-2 \cdot x}{1+x^2} \cdot y + \frac{1}{1+x^2}$ samt betingelsen: $y(1) = 2$

```
> restart
> DiffLign := y' = (-2*x)/(1+x^2)*y + 1/(1+x^2)
```

$$DiffLign := \frac{d}{dx} y(x) = -\frac{2\,x\,y(x)}{x^2+1} + \frac{1}{x^2+1} \quad \textbf{(3.1)}$$

```
> Betingelse := y(1) = 2
```

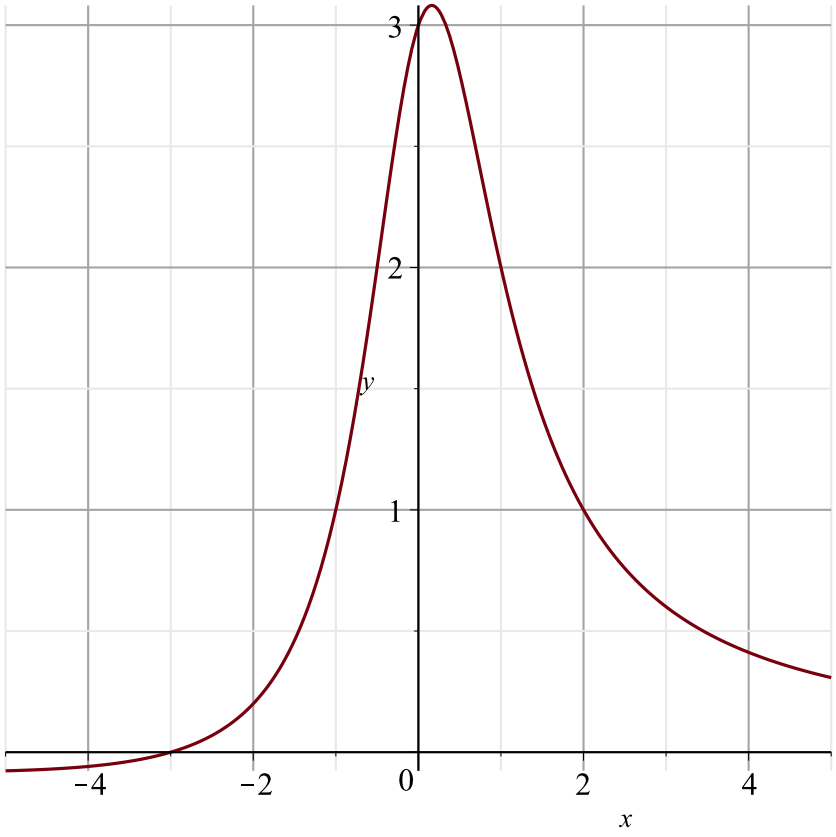
$$Betingelse := y(1) = 2 \quad \textbf{(3.2)}$$

```
> dsolve({DiffLign, Betingelse})
```

$$y(x) = \frac{x+3}{x^2+1} \quad \textbf{(3.3)}$$

Løsning: $y(x) = \frac{x+3}{1+x^2}$

```
> plot(rhs((3.3)), x=-5..5, labels=[x, y], gridlines)
```



Eksempel 51 side 52 i bog A

Differentialligning: $y' + 2 \cdot x \cdot y = x$

> *restart*

> $DiffLign := y' + 2 \cdot x \cdot y = x$

$$DiffLign := \frac{d}{dx} y(x) + 2\, x\, y(x) = x \qquad \textbf{(4.1)}$$

> $dsolve(DiffLign)$

$$y(x) = \frac{1}{2} + e^{-x^2} _C1 \qquad \textbf{(4.2)}$$

Løsning: $y(x) = \frac{1}{2} + c \cdot e^{-x^2}$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

Eksempel 52 side 53 i bog A

Differentialligning: $y' = \cos(x) \cdot y$

> *restart*

> $DiffLign := y' = \cos(x) \cdot y$

$$DiffLign := \frac{d}{dx} y(x) = \cos(x)\, y(x) \quad \textbf{(5.1)}$$

> $dsolve(DiffLign)$

$$y(x) = _C1\, e^{\sin(x)} \quad \textbf{(5.2)}$$

Fuldstændige løsning: $y(x) = c \cdot e^{\sin(x)}$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

> $Betingelse := y\left(\frac{\pi}{2}\right) = 7$

$$Betingelse := y\left(\frac{1}{2}\,\pi\right) = 7 \quad \textbf{(5.3)}$$

> $dsolve(\{DiffLign, Betingelse\})$

$$y(x) = \frac{7\, e^{\sin(x)}}{e} \quad \textbf{(5.4)}$$

> $simplify(\%)$

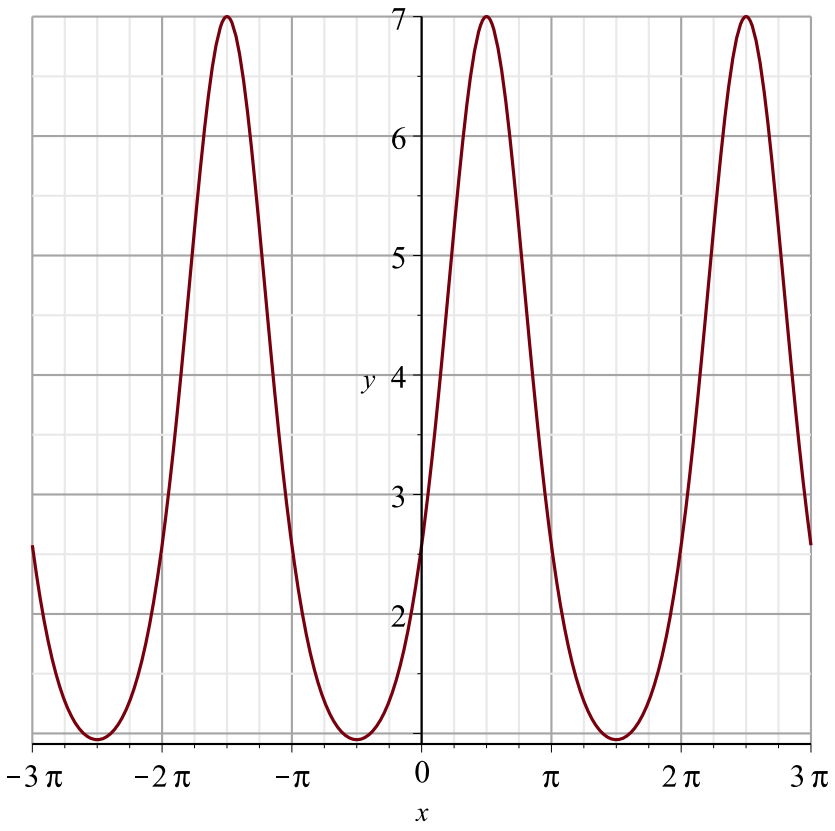
$$y(x) = 7\, e^{-1 + \sin(x)} \quad \textbf{(5.5)}$$

Partikulær løsning: $y(x) = 7 \cdot e^{\sin(x) - 1}$

> $f := x \to rhs(\textbf{(5.5)}) : f(x)$

$$7\, e^{-1 + \sin(x)} \quad \textbf{(5.6)}$$

> $plot(f(x), x = -3 \cdot \pi..3 \cdot \pi, labels = [x, y], gridlines)$



Eksempel 53 side 53 i bog A

Differentialligning: $y' - 2 \cdot y = 2$

> *restart*

> $DiffLign := y' - 2 \cdot y = 2$

$$DiffLign := \frac{d}{dx} y(x) - 2\, y(x) = 2 \qquad \textbf{(6.1)}$$

> *dsolve*(*DiffLign*)

$$y(x) = -1 + e^{2x} _C1 \qquad \textbf{(6.2)}$$

Fuldstændige løsning: $y(x) = -1 + c \cdot e^{2 \cdot x}$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

Eksempel 54 side 54 i bog A

Differentialligning: $y' - \frac{1}{x} \cdot y = 4 \cdot x^2$ for $x > 0$

> *restart*

> $DiffLign := y' - \frac{1}{x} \cdot y = 4 \cdot x^2$

$$DiffLign := \frac{d}{dx} y(x) - \frac{y(x)}{x} = 4 x^2 \quad \textbf{(7.1)}$$

> $dsolve(DiffLign)$

$$y(x) = \left(2 x^2 + _C1\right) x \quad \textbf{(7.2)}$$

> $expand(\%)$

$$y(x) = 2 x^3 + _C1\, x \quad \textbf{(7.3)}$$

Fuldstændige løsning: $y(x) = 2 \cdot x^3 + c \cdot x$, for $x > 0$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

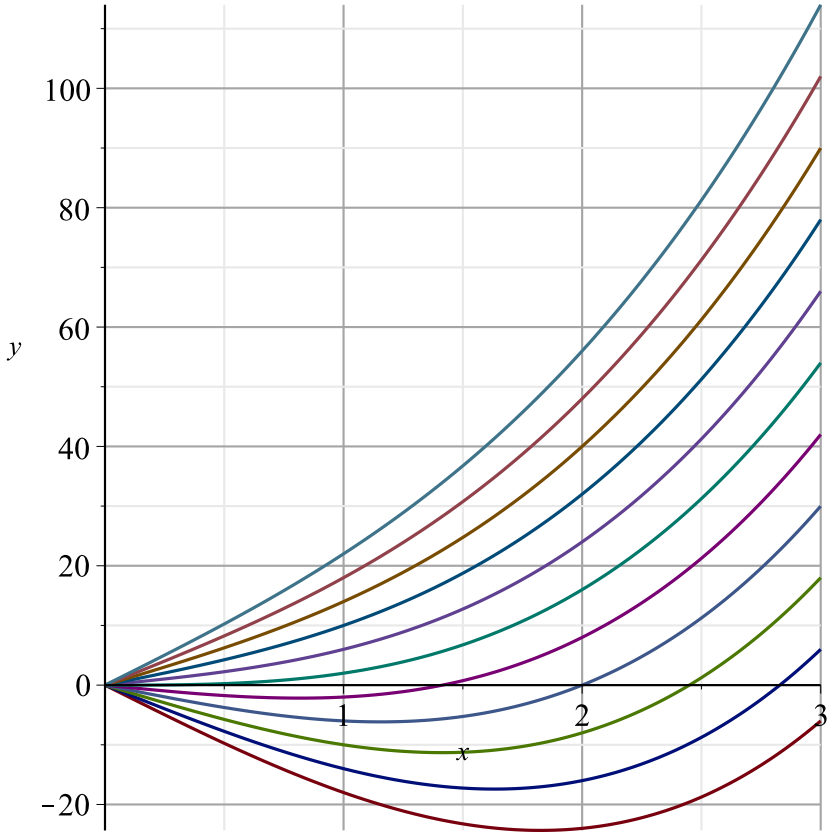
> $f := x \rightarrow rhs(\textbf{(7.3)}) : f(x)$

$$2 x^3 + _C1\, x \quad \textbf{(7.4)}$$

> $seq(f(x), _C1 = -20..20, 4)$

$$2 x^3 - 20 x,\ 2 x^3 - 16 x,\ 2 x^3 - 12 x,\ 2 x^3 - 8 x,\ 2 x^3 - 4 x,\ 2 x^3,\ 2 x^3 + 4 x,\ 2 x^3 + 8 x,\ 2 x^3 + 12 x,\ 2 x^3 + 16 x,\ 2 x^3 + 20 x \quad \textbf{(7.5)}$$

> $plot([\%], x = 0..3, labels = [x, y], gridlines)$



Eksempel 55 side 55 i bog A

Differentialligning: $y' + y = 2 \cdot e^{t}$

```
> restart
> DiffLign := y'(t) + y(t) = 2·e^t
```

$$DiffLign := D(y)(t) + y(t) = 2\,e^{t} \tag{8.1}$$

```
> dsolve(DiffLign)
```

$$y(t) = e^{t} + e^{-t}_C1 \tag{8.2}$$

Fuldstændige løsning: $y(t) = e^{t} + c \cdot e^{-t}$, hvor $c \in \mathbb{R}$ er en arbitrær konstant.

```
> Betingelse := y(0) = -6
```

$$Betingelse := y(0) = -6 \tag{8.3}$$

```
> dsolve({DiffLign, Betingelse})
```

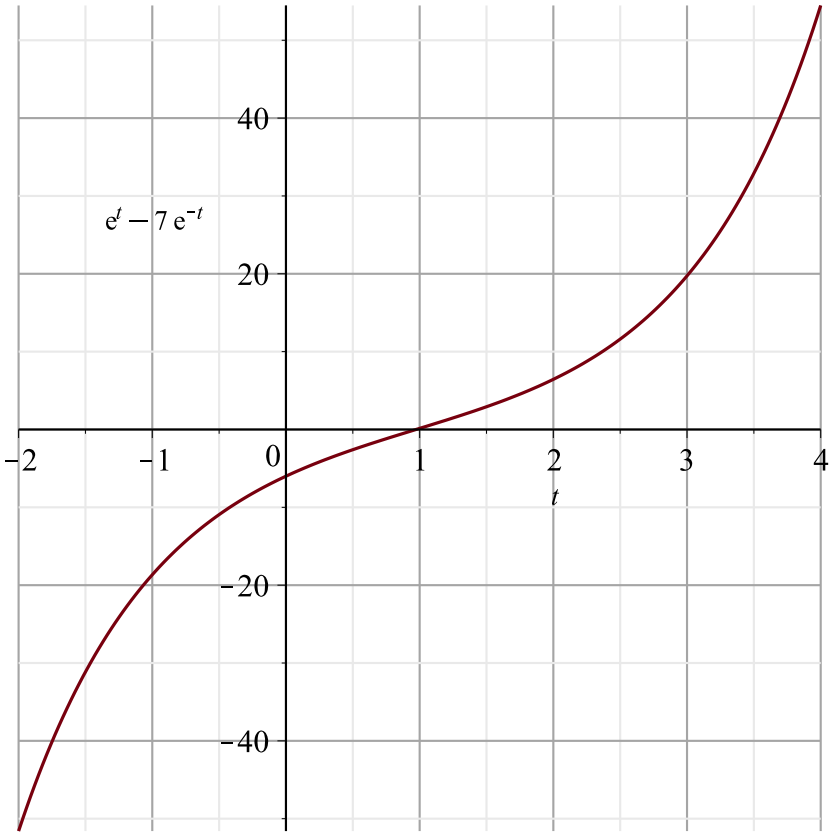
$$y(t) = e^{t} - 7\,e^{-t} \tag{8.4}$$

Partikulære løsning: $y(t) = e^{t} - 7 \cdot e^{-t}$

```
> f := t→rhs((8.4)) : f(t)
```

$$e^{t} - 7\,e^{-t} \tag{8.5}$$

```
> plot(f(t), t = -2..4, labels = [t, f(t)], gridlines)
```



Eksempel fra emathdtu-video:

youtube.com/watch?v=qJSfPxcRkPw

Differentialligning: $x'(t) + \frac{2}{t} \cdot x(t) = 8 \cdot t - \frac{10}{t}$, hvor $t > 0$

```
> restart
```

```
> DiffLign := x'(t) + 2/t·x(t) = 8·t − 10/t
```

$$DiffLign := \mathrm{D}(x)(t) + \frac{2\,x(t)}{t} = 8\,t - \frac{10}{t} \qquad \textbf{(9.1)}$$

```
> dsolve(DiffLign)
```

$$x(t) = -5 + 2\,t^2 + \frac{-C1}{t^2} \qquad \textbf{(9.2)}$$

Fuldstændig løsning: $x(t) = 2 \cdot t^2 - 5 + \frac{c}{t^2}$, for $t > 0$, og hvor $c \in \mathbb{R}$ er en arbitrær konstant.

```
> Betingelse := x(1) = 2
```

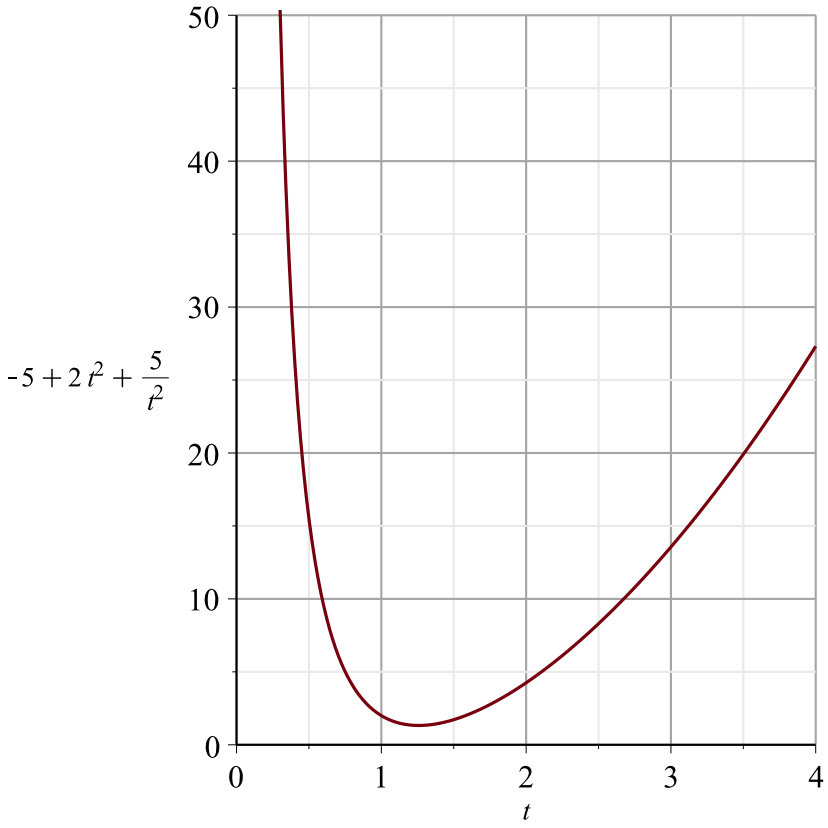
$$Betingelse := x(1) = 2 \qquad \textbf{(9.3)}$$

```
> dsolve( {DiffLign, Betingelse} )
```

$$x(t) = -5 + 2\,t^2 + \frac{5}{t^2} \qquad \textbf{(9.4)}$$

Partikulær løsning: $x(t) = 2 \cdot t^2 - 5 + \frac{5}{t^2}$, hvor $t > 0$

```
> f := t→rhs(  (9.4)  ) : f(t)
```

$$-5 + 2\,t^2 + \frac{5}{t^2} \qquad \textbf{(9.5)}$$

```
> plot( f(t), t = 0 ..4, f = 0 ..50, labels = [t, f(t) ], gridlines)
```