Differentialligninger
Logistisk vækst

$y' = y \cdot (b - a \cdot y)$

> restart

> $DiffLign := y' = y \cdot (b - a \cdot y)$

$$DiffLign := \frac{d}{dx} y(x) = y(x)\ (b - a\, y(x)) \quad \textbf{(1.1)}$$

> $dsolve(DiffLign)$

$$y(x) = \frac{b}{e^{-b\,x}\ _C1\ b + a} \quad \textbf{(1.2)}$$

Normalt forkorter man med a i tæller og nævner. Konstanten foran $e^{-b \cdot x}$ i nævneren kaldes så c.
Konklusion: den fuldstændige løsning til differentialligningen $y' = y \cdot (b - a \cdot y)$ er

$$y = \frac{\frac{b}{a}}{1 + c \cdot e^{-b \cdot x}} \text{ hvor } c \in \mathbb{R}_+$$

> $f := x \to subs(_C1 = c, rhs(\textbf{(1.2)}))\ : 'f(x)' = f(x)$

$$f(x) = \frac{b}{e^{-b\,x}\ c\ b + a} \quad \textbf{(1.3)}$$

Eksempel på løsningskurve:

> $a := 3; b := 2; c := 1$

$$a := 3$$
$$b := 2$$
$$c := 1 \quad \textbf{(1.4)}$$

> $f(x)$

$$\frac{2}{2\ e^{-2\,x} + 3} \quad \textbf{(1.5)}$$

> $with(plots) : with(DEtools) :$

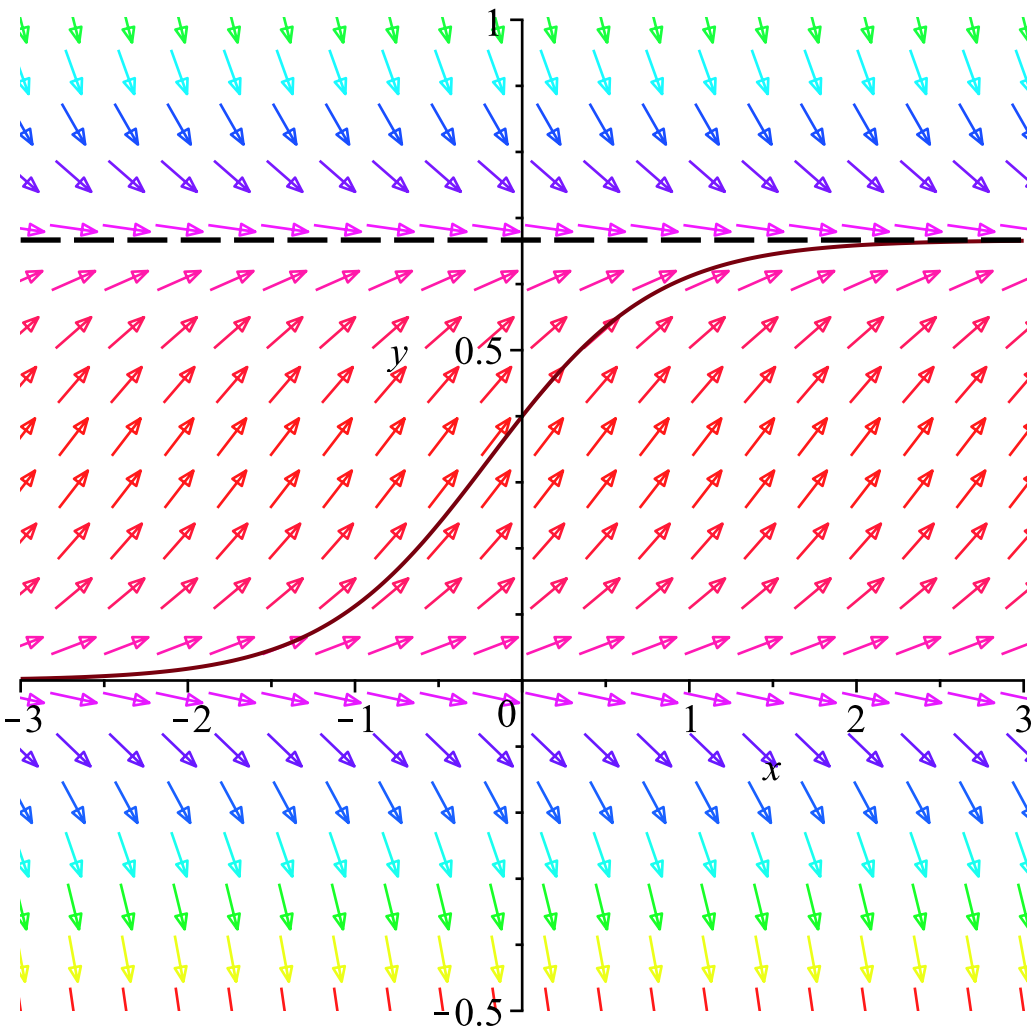
> $FeltFarve := dfieldplot\left(DiffLign, y(x), x = -3..3, y = -0.5..\frac{3}{2} \cdot \frac{b}{a}, color = rhs(DiffLign), arrows = SLIM \right) :$

> $FeltSort := dfieldplot\left(DiffLign, y(x), x = -3..3, y = -0.5..\frac{3}{2} \cdot \frac{b}{a}, color = grey, arrows = SLIM \right) :$

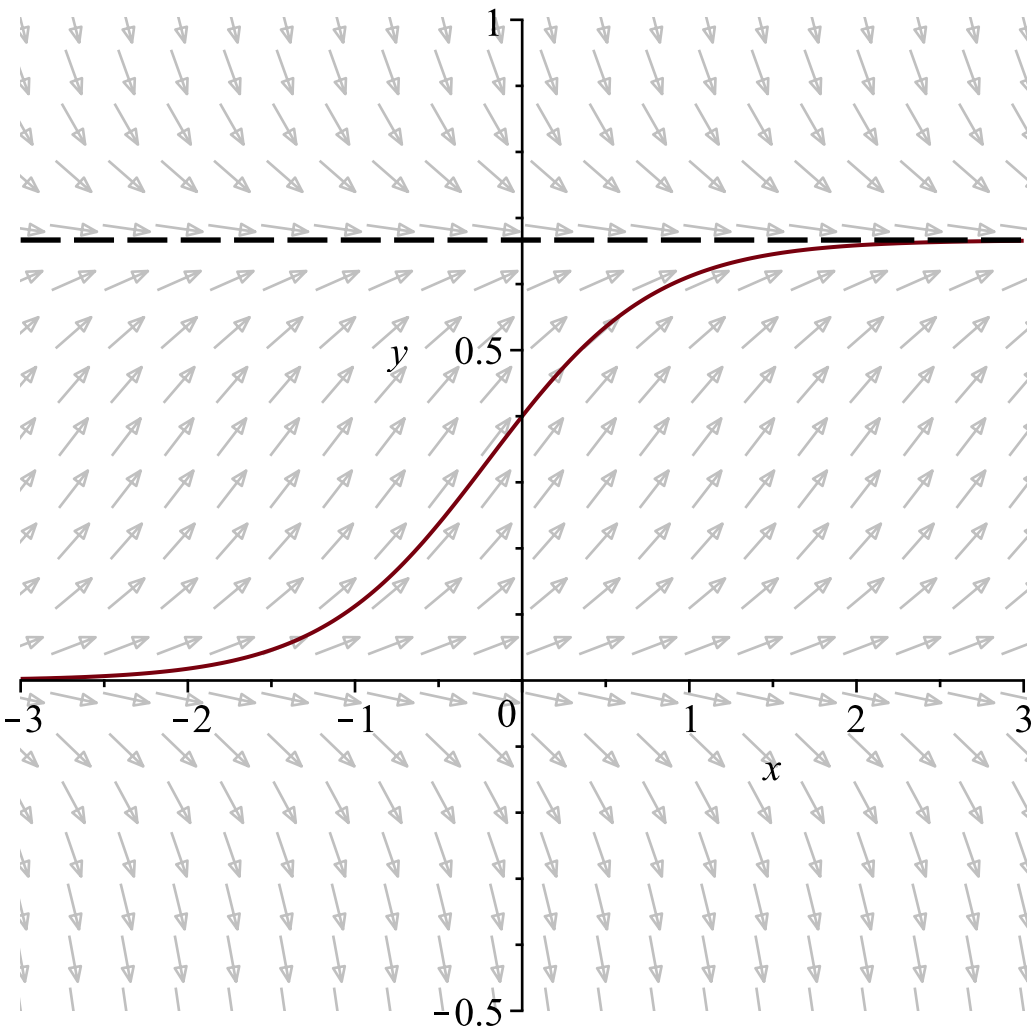
> $Kurve := plot(f(x), x = -3..3) :$

> $Stabil := plot\left(\frac{b}{a}, x = -3..3, color = black, linestyle = dash, thickness = 2 \right) :$

> $display(FeltFarve, Kurve, Stabil, view = [-3..3, -0.5..1], labels = [x, y])$




```
> display(FeltSort, Kurve, Stabil, view = [ -3 ..3, -0.5 ..1 ], labels = [ x, y ])
```



NB: $f(x)=\frac{b}{a}$ er en **stabil** løsning, hvorimod $f(x)=0$ er en **ustabil** løsning.

$y'=a\cdot y\cdot(M-y)$

```
> restart
> DiffLign := y'=a·y·(M−y)
```

$$DiffLign := \frac{d}{dx}y(x)=a\,y(x)\,(M-y(x)) \tag{2.1}$$

```
> dsolve(DiffLign)
```

$$y(x)=\frac{M}{1+e^{-aMx}_C1\,M} \tag{2.2}$$

Konklusion: den fuldstændige løsning til differentialligningen $y'=y\cdot(b-a\cdot y)$ er $y=\frac{M}{1+c\cdot e^{-a\cdot M\cdot x}}$ hvor $c\in\mathbb{R}_+$

```
> f := x→subs(_C1=c, rhs((2.2))) :'f(x)'=f(x)
```

$$f(x)=\frac{M}{1+e^{-aMx}\,c\,M} \tag{2.3}$$

Eksempel på løsningskurve:

```
> a := 3; M := 2; c := 1
```

$$a := 3$$
$$M := 2$$
$$c := 1 \tag{2.4}$$

> $f(x)$

$$\frac{2}{1 + 2\,e^{-6x}} \tag{2.5}$$

> $with(plots) : with(DEtools) :$

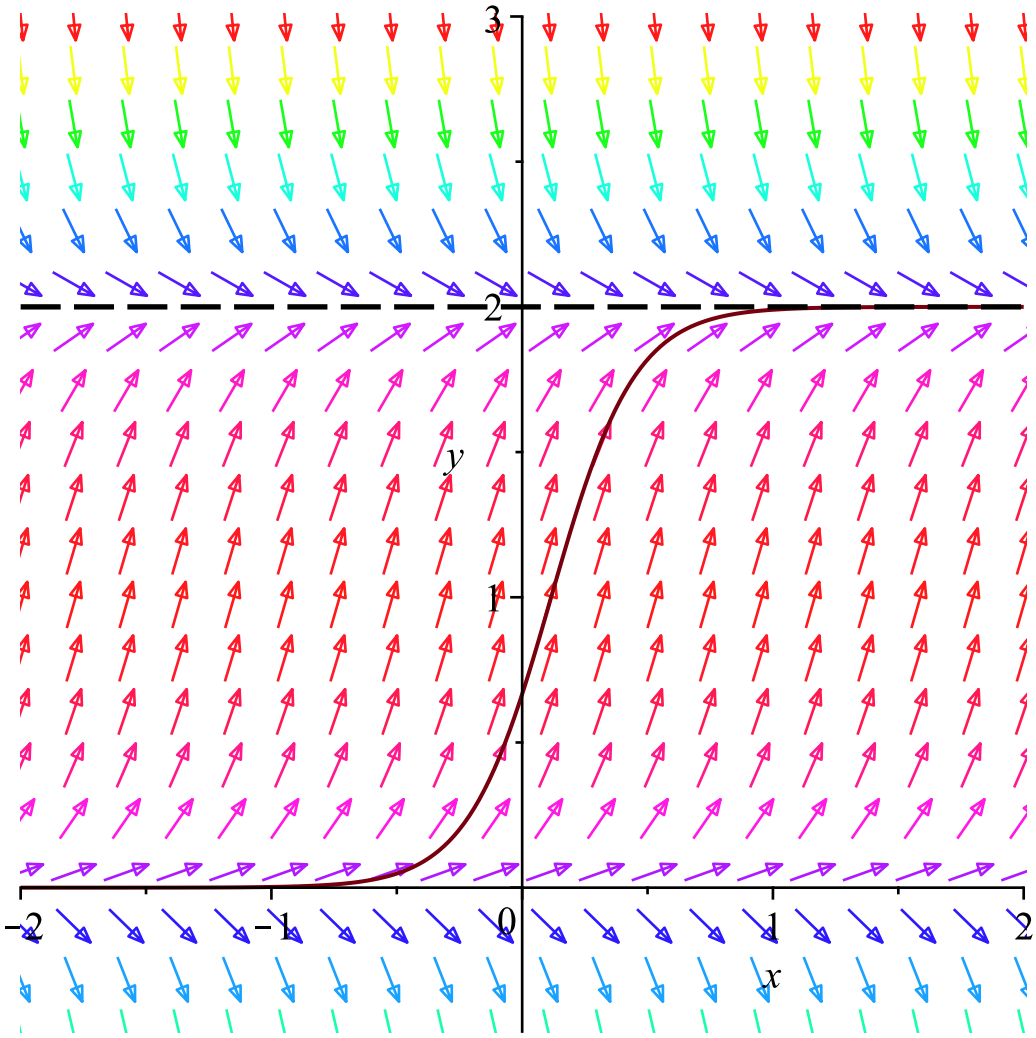
> $FeltFarve := dfieldplot\left(DiffLign, y(x), x = -2..2, y = -0.5..\frac{3}{2}\cdot M, color = rhs(DiffLign), arrows = SLIM\right) :$

> $FeltSort := dfieldplot\left(DiffLign, y(x), x = -2..2, y = -0.5..\frac{3}{2}\cdot M, color = grey, arrows = SLIM\right) :$

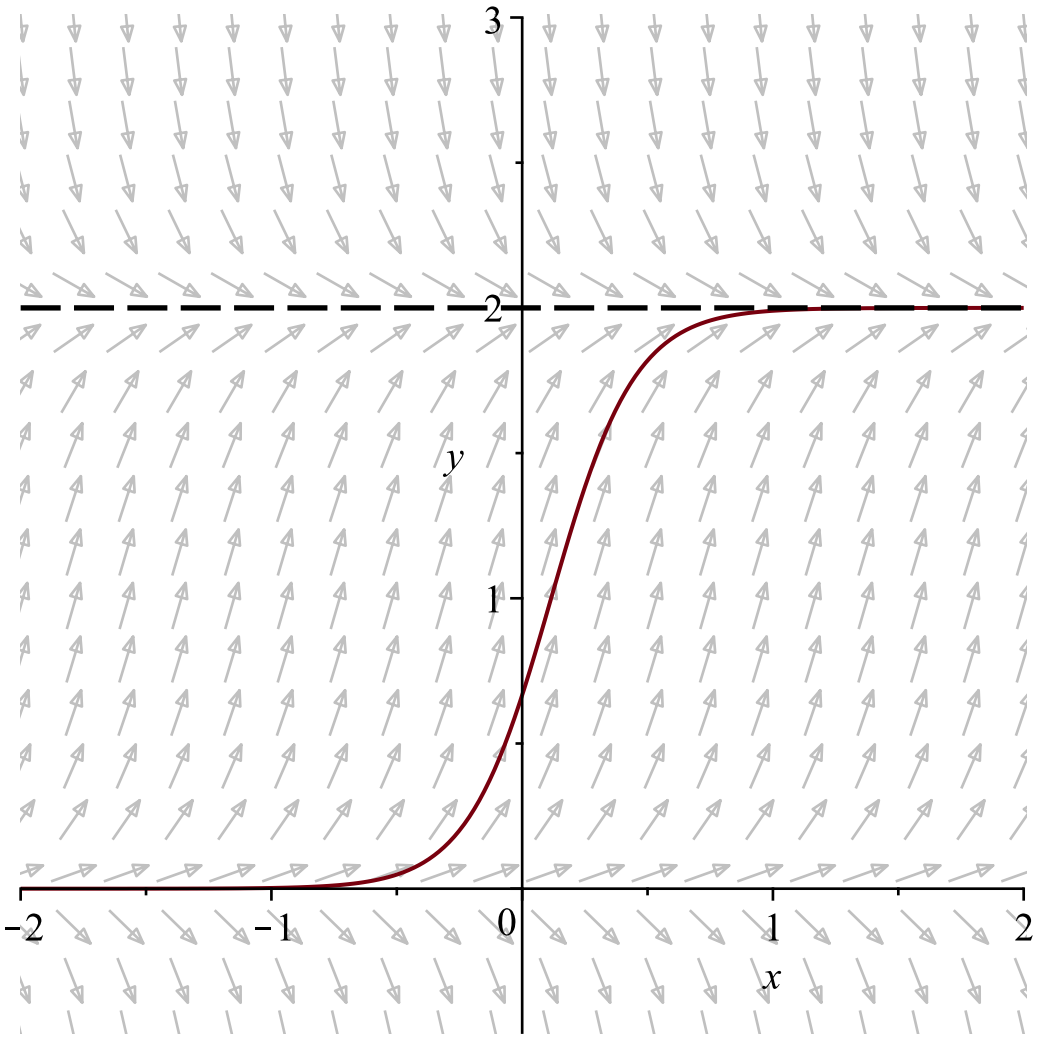
> $Kurve := plot(f(x), x = -2..2) :$

> $Stabil := plot(M, x = -2..2, color = black, linestyle = dash, thickness = 2) :$

> $display\left(FeltFarve, Kurve, Stabil, view = \left[-2..2, -0.5..\frac{3}{2}\cdot M\right], labels = [x, y]\right)$



> $display\left(FeltSort, Kurve, Stabil, view = \left[-2..2, -0.5..\frac{3}{2}\cdot M\right], labels = [x, y]\right)$



NB: f(x) = M er en **stabil** løsning, hvorimod $f(x) = 0$ er en **ustabil** løsning.

Eksempel 45 side 46 i bog A

> *restart*

> $a := 2\cdot 10^{-5}; M := 1500$

$$a := \frac{1}{50000}$$
$$M := 1500 \quad \textbf{(3.1)}$$

> $DiffLign := y' = a\cdot y\cdot (M - y)$

$$DiffLign := \frac{d}{dx}\,y(x) = \frac{y(x)\,(1500 - y(x))}{50000} \quad \textbf{(3.2)}$$

> $Betingelse := y(0) = 50$

$$Betingelse := y(0) = 50 \quad \textbf{(3.3)}$$

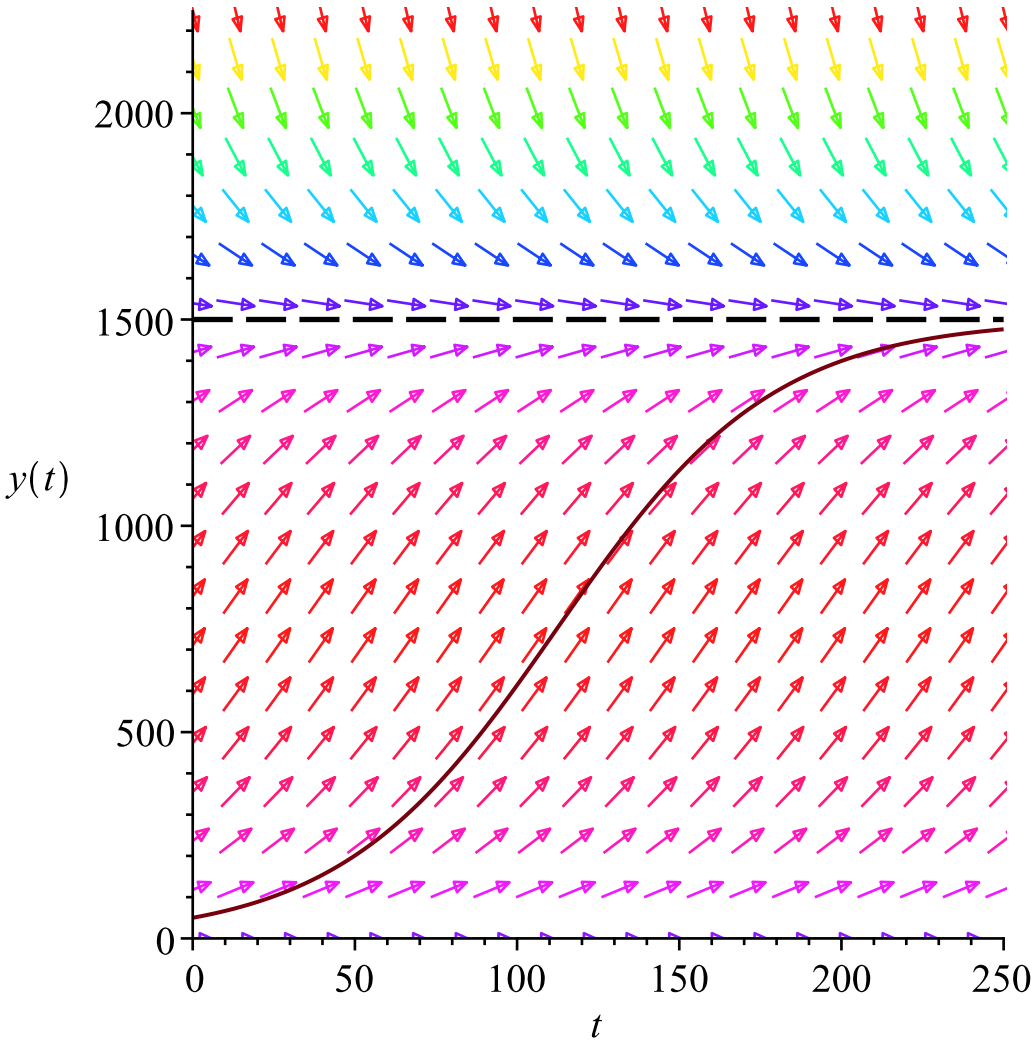
> $dsolve(\{DiffLign, Betingelse\})$

$$y(x) = \frac{1500}{1 + 29\,e^{-\frac{3x}{100}}} \quad \textbf{(3.4)}$$

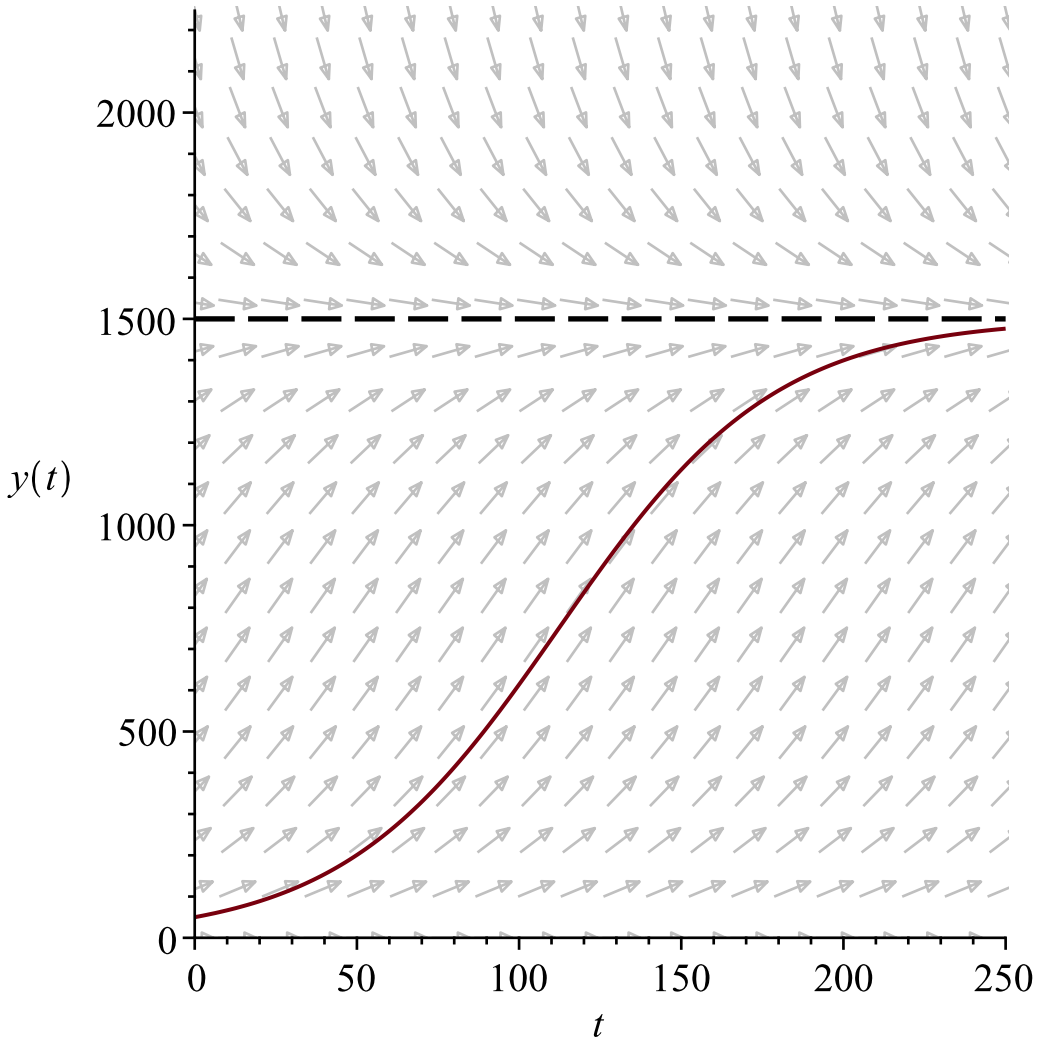
> $f := x \to rhs(\textbf{(3.4)}) : 'f(x)' = f(x)$

$$f(x) = \frac{1500}{1 + 29\, e^{-\frac{3x}{100}}} \qquad \textbf{(3.5)}$$

```
> with(plots) : with(DEtools) :
> FeltFarve := dfieldplot( DiffLign, y(x), x = 0 ..250, y = -0.5 .. 3/2 ·M, color
      = rhs(DiffLign) , arrows = SLIM ) :
> FeltSort := dfieldplot( DiffLign, y(x), x = 0 ..250, y = -0.5 .. 3/2 ·M, color = grey, arrows
      = SLIM ) :
> Kurve := plot( f(x), x = 0 ..250) :
> Stabil := plot(M, x = 0 ..250, color = black, linestyle = dash, thickness = 2) :
> display( FeltFarve, Kurve, Stabil, view = [0 ..250, 0 .. 3/2 · M], labels = [t, y(t) ] )
```




```
> display( FeltSort, Kurve, Stabil, view = [0 ..250, 0 .. 3/2 · M], labels = [t, y(t) ] )
```



Eksempel 46 side 47 i bog A

```
> restart
> a := 1; M := 2
```

$$a := 1$$
$$M := 2 \tag{4.1}$$

```
> DiffLign := y'=a·y·(M−y)
```

$$DiffLign := \frac{d}{dx} y(x) = y(x)\,(2 - y(x)) \tag{4.2}$$

```
> Betingelse := y(0) = 1/2
```

$$Betingelse := y(0) = \frac{1}{2} \tag{4.3}$$

```
> dsolve({DiffLign, Betingelse})
```

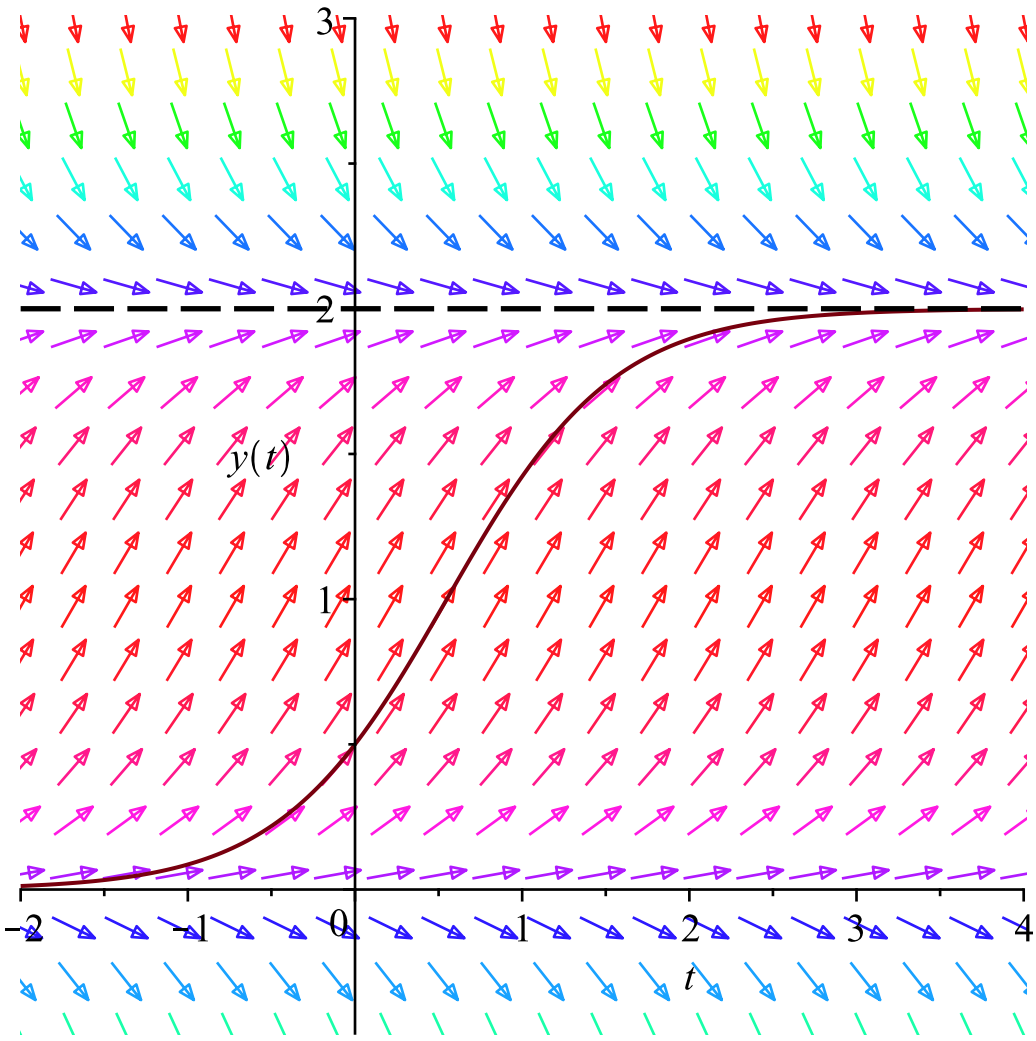
$$y(x) = \frac{2}{1 + 3\,e^{-2x}} \tag{4.4}$$

```
> f := x→rhs((4.4)) :f(x)'=f(x)
```

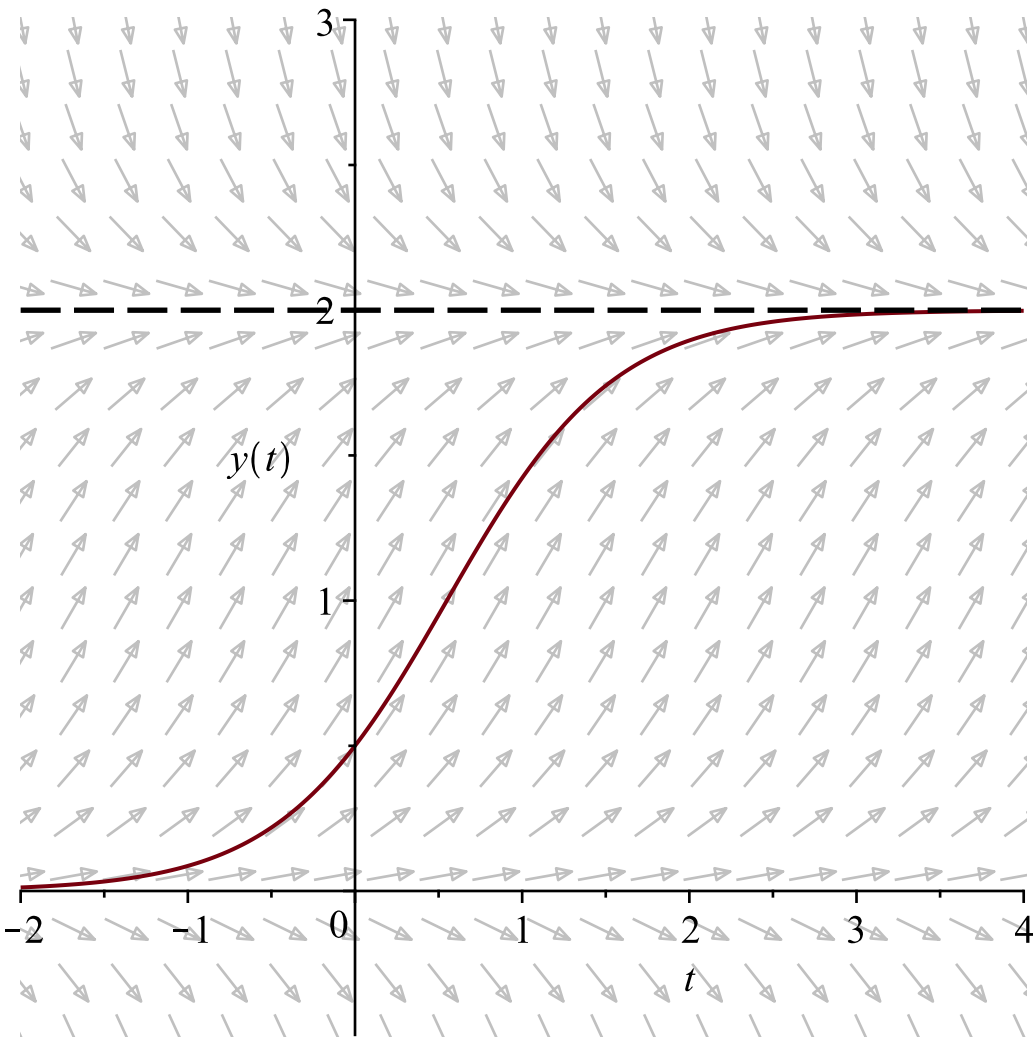
$$f(x) = \frac{2}{1 + 3\,e^{-2x}} \tag{4.5}$$

```
> with(plots) : with(DEtools) :
```

```
> FeltFarve := dfieldplot(DiffLign, y(x), x = -2..4, y = -0.5..(3/2)·M, color = rhs(DiffLign),
      arrows = SLIM):
> FeltSort := dfieldplot(DiffLign, y(x), x = -2..4, y = -0.5..(3/2)·M, color = grey, arrows
      = SLIM):
> Kurve := plot(f(x), x = -2..4):
> Stabil := plot(M, x = -2..4, color = black, linestyle = dash, thickness = 2):
> display(FeltFarve, Kurve, Stabil, view = [-2..4, -0.5..(3/2)·M], labels = [t, y(t)])
```




```
> display(FeltSort, Kurve, Stabil, view = [-2..4, -0.5..(3/2)·M], labels = [t, y(t)])
```



Eksempel 47 og 48 side 48 og 49 i bog A

> *restart*

> $a := 0.000001;\ b := 0.5$

$$a := 1.\ 10^{-6}$$
$$b := 0.5 \qquad \textbf{(5.1)}$$

> $DiffLign := y' = y \cdot (b - a \cdot y)$

$$DiffLign := \frac{d}{dx}\, y(x) = y(x)\ (0.5 - 1.\ 10^{-6}\, y(x)) \qquad \textbf{(5.2)}$$

> $Betingelse := y(0) = 10000$

$$Betingelse := y(0) = 10000 \qquad \textbf{(5.3)}$$

> $dsolve(\{DiffLign, Betingelse\})$

$$y(x) = \frac{500000}{1 + 49\, e^{-\frac{x}{2}}} \qquad \textbf{(5.4)}$$

> $f := x \rightarrow rhs(\textbf{(5.4)}) :'f(x)' = f(x)$

$$f(x) = \frac{500000}{1 + 49\, e^{-\frac{x}{2}}} \qquad \textbf{(5.5)}$$

> $f'(x)$

$$\frac{12250000\, e^{-\frac{x}{2}}}{\left(1+49\, e^{-\frac{x}{2}}\right)^2} \tag{5.6}$$

Populationen vokser hurtigst, når $f'(x)$ er maksimal, eller når $f(x)$ er halvdelen af maksimum.

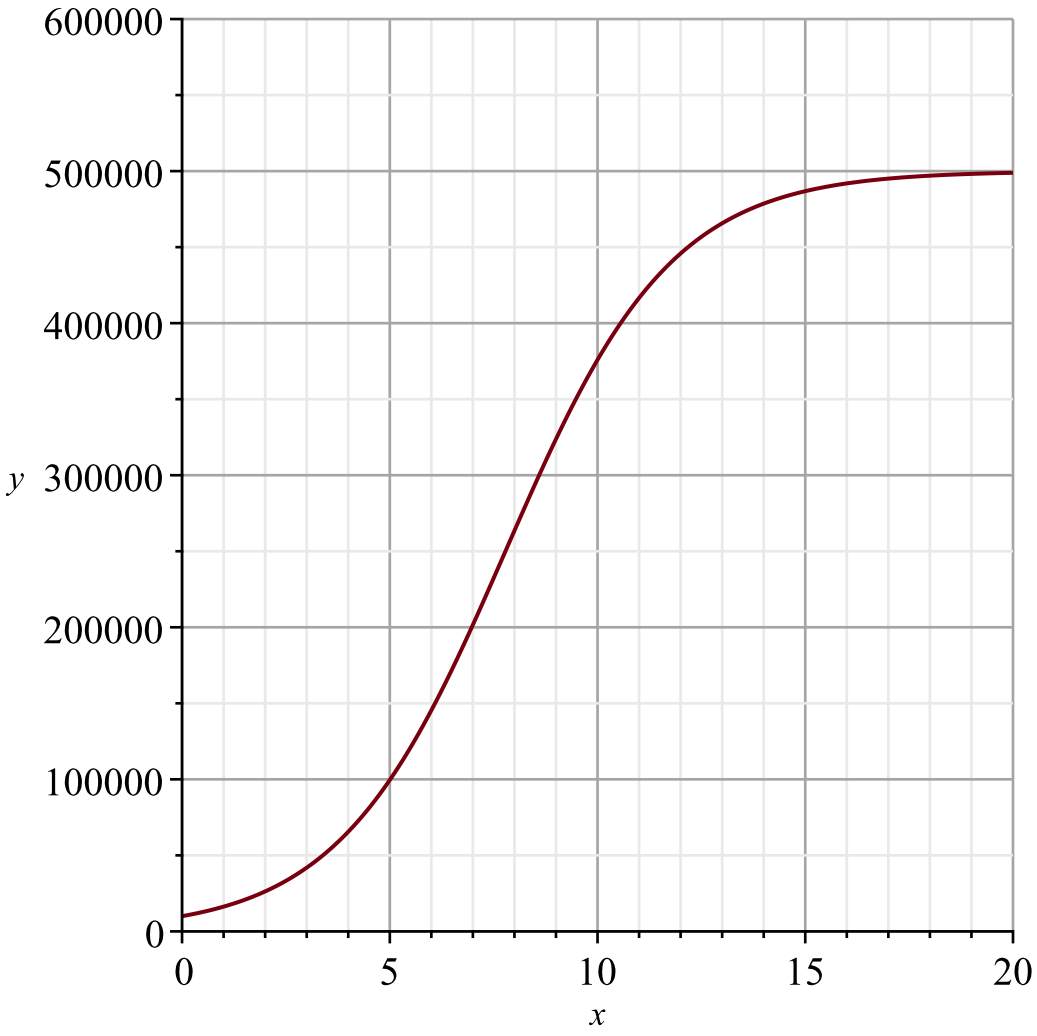
> $maximize(f'(x), x, x > 0, location); evalf(\%)$

$$62500,\ \{[\{x = 4\ln(7)\}, 62500]\}$$
$$62500.,\ \{[\{x = 7.783640596\}, 62500.]\} \tag{5.7}$$

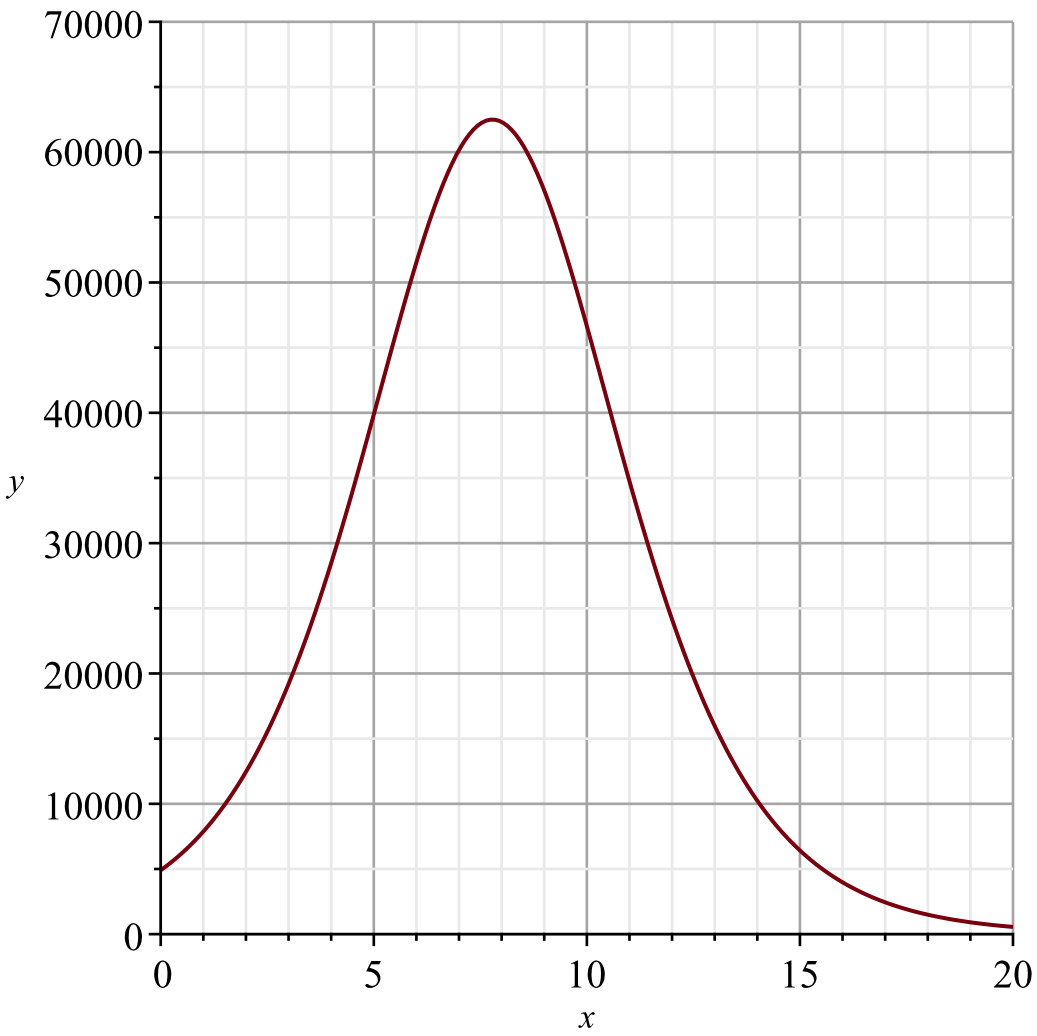
> $solve\left(f(x) = \frac{1}{2}\cdot\frac{b}{a}, x\right)$

$$7.783640596 \tag{5.8}$$

> $plot(f(x), x = 0..20, y = 0..600000, gridlines)$



> $plot(f'(x), x = 0..20, y = 0..70000, gridlines)$



Konklusion: Løsningen er $f(x) = \frac{500000}{1 + 49\, e^{-\frac{1}{2} t}}$ og populationen vokser mest, når $t \approx 7.8$