Analytisk funktion

En C^{∞} – funktion, hvis Taylor-række konvergerer imod funktionen i en omegn af punktet x_0, siges at være en **analytisk funktion**.
Læs nærmere: **http://en.wikipedia.org/wiki/Analytic_function**

Ikke alle funktion, som kan differentieres uendelig mange gange er analytiske!

Modeksempel:

Definer følgende funktion:

$$f(x) = \begin{cases} 0 & \text{for } x \leq 0 \\ e^{-\frac{1}{x^2}} & \text{for } x > 0 \end{cases}$$

Denne funktion $f \in C^{\infty}$, men er IKKE en analytisk funktion, da Taylor-rækken ud fra ud fra $x_0 = 0$ er identisk med 0 !

> *restart*

Venstre del af funktionen:

> $fv := x \to 0$

$$fv := x \to 0 \tag{1}$$

Højre del af funktionen:

> $fh := x \to \exp\left(-\frac{1}{x^2}\right)$

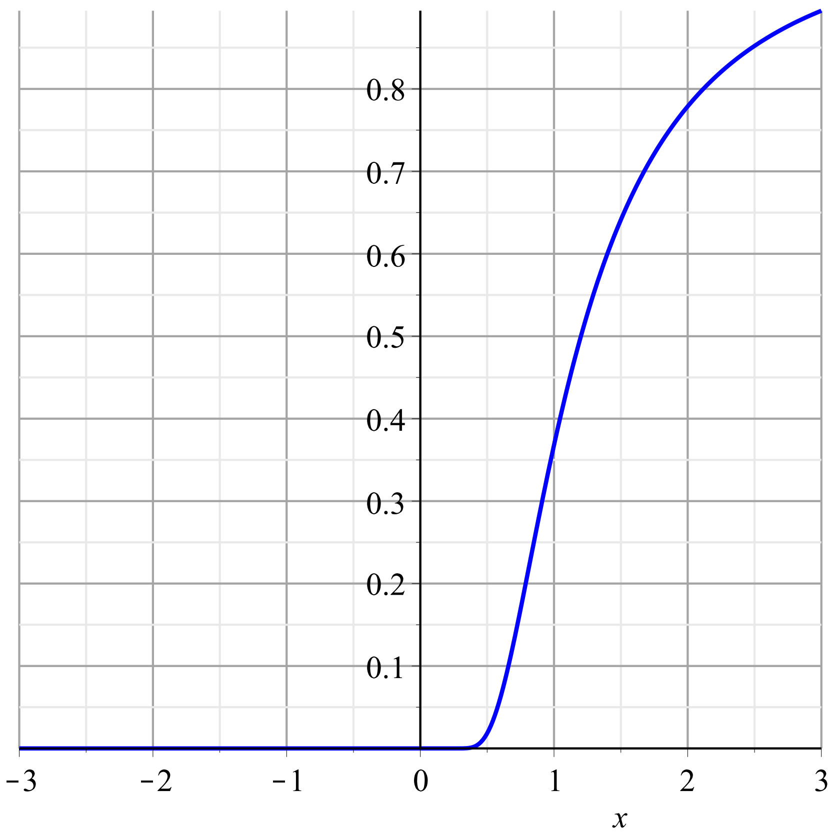
$$fh := x \to e^{-\frac{1}{x^2}} \tag{2}$$

> *with(plots) :*

> *plot1 := plot(fv(x), x =−3 ..0, color = blue, thickness = 2) :*

> *plot2 := plot(fh(x), x = 0 ..3, color = blue, thickness = 2) :*

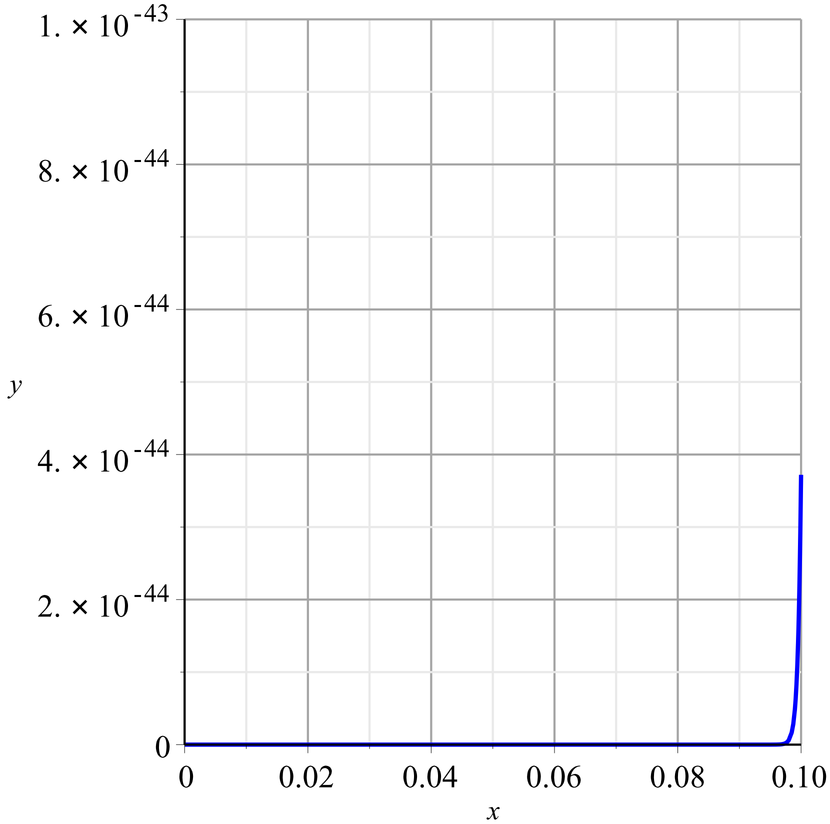
> *display(plot1, plot2, gridlines)*



> $fh(0.1)$

$$3.720075976\ 10^{-44} \tag{3}$$

> $plot\left(fh(x), x = 0 ..0.1, y = 0 ..10^{-43}, color = blue, thickness = 2, gridlines\right)$



Funktionen er så flad i $x_0 = 0$, at alle afledede i dette punkt er 0!

Derfor er Taylorrækken ud fra $x_0 = 0$ identisk med nulpolynomiet.
Det betyder, at Taylorrækken ud fra $x_0 = 0$ ALDRIG konvergerer imod $f(x)$ for noget $x > 0$.

> $\lim_{x \to 0+} (fh(x))$

$$0 \tag{4}$$

Dvs. $f(0) = 0$, og f er kontinuert i 0.

> $f_1 := unapply\left(\frac{d}{dx} fh(x), x\right)$

$$f_1 := x \to \frac{2\, e^{-\frac{1}{x^2}}}{x^3} \tag{5}$$

> $\lim_{x \to 0+} \left(f_1(x)\right)$

$$0 \tag{6}$$

> $\lim_{x \to 0+} \left(\frac{fh(x) - 0}{x - 0} \right)$

$$0 \qquad \textbf{(7)}$$

Dvs. $f'(0) = 0$, og f' er kontinuert i 0.

> $f_2 := unapply\left(\frac{d}{dx} f_1(x), x \right)$

$$f_2 := x \to -\frac{6\,e^{-\frac{1}{x^2}}}{x^4} + \frac{4\,e^{-\frac{1}{x^2}}}{x^6} \qquad \textbf{(8)}$$

> $\lim_{x \to 0+} \left(f_2(x) \right)$

$$0 \qquad \textbf{(9)}$$

> $\lim_{x \to 0+} \left(\frac{f_1(x) - 0}{x - 0} \right)$

$$0 \qquad \textbf{(10)}$$

Dvs. $f''(0) = 0$, og f'' er kontinuert i 0.

Osv.

> **for** n **from** 2 **by** 1 **to** 10 **do** $f_{n+1} := unapply\left(\frac{d}{dx} f_n(x), x \right)$ **end do**

$$f_3 := x \to \frac{24\,e^{-\frac{1}{x^2}}}{x^5} - \frac{36\,e^{-\frac{1}{x^2}}}{x^7} + \frac{8\,e^{-\frac{1}{x^2}}}{x^9}$$

$$f_4 := x \to -\frac{120\,e^{-\frac{1}{x^2}}}{x^6} + \frac{300\,e^{-\frac{1}{x^2}}}{x^8} - \frac{144\,e^{-\frac{1}{x^2}}}{x^{10}} + \frac{16\,e^{-\frac{1}{x^2}}}{x^{12}}$$

$$f_5 := x \to \frac{720\,e^{-\frac{1}{x^2}}}{x^7} - \frac{2640\,e^{-\frac{1}{x^2}}}{x^9} + \frac{2040\,e^{-\frac{1}{x^2}}}{x^{11}} - \frac{480\,e^{-\frac{1}{x^2}}}{x^{13}} + \frac{32\,e^{-\frac{1}{x^2}}}{x^{15}}$$

$$f_6 := x \to -\frac{5040\,e^{-\frac{1}{x^2}}}{x^8} + \frac{25200\,e^{-\frac{1}{x^2}}}{x^{10}} - \frac{27720\,e^{-\frac{1}{x^2}}}{x^{12}} + \frac{10320\,e^{-\frac{1}{x^2}}}{x^{14}} - \frac{1440\,e^{-\frac{1}{x^2}}}{x^{16}} + \frac{64\,e^{-\frac{1}{x^2}}}{x^{18}}$$

$$f_7 := x \to \frac{40320\,e^{-\frac{1}{x^2}}}{x^9} - \frac{262080\,e^{-\frac{1}{x^2}}}{x^{11}} + \frac{383040\,e^{-\frac{1}{x^2}}}{x^{13}} - \frac{199920\,e^{-\frac{1}{x^2}}}{x^{15}} + \frac{43680\,e^{-\frac{1}{x^2}}}{x^{17}}$$

$$-\frac{4032\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{19}}+\frac{128\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{21}}$$

$$f_8 := x \to -\frac{362880\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{10}}+\frac{2963520\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{12}}-\frac{5503680\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{14}}+\frac{3764880\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{16}}$$
$$-\frac{1142400\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{18}}+\frac{163968\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{20}}-\frac{10752\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{22}}+\frac{256\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{24}}$$

$$f_9 := x \to \frac{3628800\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{11}}-\frac{36288000\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{13}}+\frac{82978560\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{15}}-\frac{71245440\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{17}}$$
$$+\frac{28092960\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{19}}-\frac{5564160\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{21}}+\frac{564480\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{23}}-\frac{27648\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{25}}+\frac{512\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{27}}$$

$$f_{10} := x \to -\frac{39916800\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{12}}+\frac{479001600\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{14}}-\frac{1317254400\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{16}}+\frac{1377129600\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{18}}$$
$$-\frac{676257120\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{20}}+\frac{173033280\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{22}}-\frac{24111360\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{24}}+\frac{1820160\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{26}}$$
$$-\frac{69120\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{28}}+\frac{1024\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{30}}$$

$$f_{11} := x \to \frac{479001600\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{13}}-\frac{6785856000\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{15}}+\frac{22034073600\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{17}} \quad \textbf{(11)}$$
$$-\frac{27422841600\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{19}}+\frac{16279401600\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{21}}-\frac{5159246400\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{23}}+\frac{924739200\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{25}}$$
$$-\frac{95546880\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{27}}+\frac{5575680\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{29}}-\frac{168960\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{31}}+\frac{2048\,\mathrm{e}^{-\frac{1}{x^2}}}{x^{33}}$$

> **for** n **from** 2 **by** 1 **to** 10 **do** $\lim_{x \to 0+} \left(f_{n+1}(x)\right)$ **end do**

0
0
0
0
0

$$0$$
$$0$$
$$0$$
$$0 \qquad (12)$$

> **for** n **from** 2 **by** 1 **to** 10 **do** $\lim_{x \to 0^+} \left(\frac{f_n(x) - 0}{x - 0} \right)$ **end do**

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0 \qquad (13)$$