Uge 02, F15, Opgave 5: Kædereglen

**Opgaven er her løst ved brug af *dsolve* til direkte at bestemme de 2 løsninger til differentialligningen,
og så indsætte løsningerne i *f* førend der differentieres og indsættes 0.**

Om de specielle funktioner, som optræder nedenfor:

BesselJ og *BesselY*: http://www.maplesoft.com/support/help/Maple/view.aspx?path=Bessel

Gamma: http://www.maplesoft.com/support/help/Maple/view.aspx?path=GAMMA

```
> restart
> DiffLign := w'(u) = w(u)^2 + u^2
```

$$DiffLign := \mathrm{D}(w)(u) = w(u)^2 + u^2 \tag{1}$$

```
> dsolve({DiffLign, w(0) = 1})
```

$$w(u) = \tag{2}$$

$$\begin{cases} \dfrac{\left(\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} & u < 0 \\ 1 & u = 0 \\ \dfrac{\left(-\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right)u}{\dfrac{\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} & 0 < u \end{cases}$$

```
> x := unapply(rhs(%), u)
```

(3)

$$x := u \to piecewise\left(u < 0, \frac{\left(\frac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2} u^2 \right) \left(\Gamma\left(\frac{3}{4} \right)^2 + \pi \right)}{\Gamma\left(\frac{3}{4} \right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2} u^2 \right) \right) u}{-\frac{\left(\Gamma\left(\frac{3}{4} \right)^2 + \pi \right) \mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2} u^2 \right)}{\Gamma\left(\frac{3}{4} \right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2} u^2 \right)}, u = 0, 1, 0 < u, \frac{\left(-\frac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2} u^2 \right) \left(-\Gamma\left(\frac{3}{4} \right)^2 + \pi \right)}{\Gamma\left(\frac{3}{4} \right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2} u^2 \right) \right) u}{\frac{\left(-\Gamma\left(\frac{3}{4} \right)^2 + \pi \right) \mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2} u^2 \right)}{\Gamma\left(\frac{3}{4} \right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2} u^2 \right)} \right) \quad \textbf{(3)}$$

> $dsolve(\{DiffLign, w(0) = 2\})$

(4)

$$w(u) = \begin{cases} \dfrac{\left(\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} & u < 0 \\ 2 & u = 0 \\ \dfrac{\left(\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} & 0 < u \end{cases} \quad \textbf{(4)}$$

> $y := unapply(rhs(\%), u)$

$$y := u \to piecewise\left(u < 0, \dfrac{\left(\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}, u = 0, 2, 0\right) \quad \textbf{(5)}$$

$$< u, \frac{\left(\frac{\text{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \text{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right) u}{-\frac{\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)\text{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \text{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} >$$

> $f := (x, y) \to \ln(1 + x \cdot y^2) : f(x, y)$

$$\ln(x\,y^2 + 1) \tag{6}$$

> $f2 := u \to f(x(u), y(u)) : f2(u)$

$$\ln\left(1 + \left(\right.\right. \tag{7}$$

$$\begin{cases} \dfrac{\left(\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & u<0 \\ 1 & u=0 \\ \dfrac{\left(-\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{\dfrac{\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & 0<u \end{cases}$$

$$\left\{ \begin{array}{ll} \dfrac{\left(\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right) u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} & u < 0 \\ 2 & u = 0 \\ \dfrac{\left(\dfrac{\mathrm{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \mathrm{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right) u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)\mathrm{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \mathrm{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} & 0 < u \end{array} \right.$$

$$2$$

> $f2'(u)$

$$\left\{\left\{\left\{\left\{\frac{\left(\left(-\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)-\frac{3}{2}\,\frac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)}{u^2}\right)u\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\left(-\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-\frac{\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-\frac{\left(-\left(\left(-\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)-\frac{3}{2}\,\frac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)}{u^2}\right)u\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\left(-\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)\right.\right.\right.\right.$$

$$\left.\left.\left.\left.-\frac{\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)\right.\right.\right.\right.$$

$$\left\{ \frac{\left(\frac{\text{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \text{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right) u}{-\frac{\left(\Gamma\left(\frac{3}{4}\right)^2 + 2\pi\right)\text{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \text{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} \quad u < 0 \right.$$

$$\left. 2 \quad u = 0 \right.$$

$$\left. \frac{\left(\frac{\text{BesselJ}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2} - \text{BesselY}\left(-\frac{3}{4}, \frac{1}{2}u^2\right)\right) u}{-\frac{\left(\Gamma\left(\frac{3}{4}\right)^2 - 2\pi\right)\text{BesselJ}\left(\frac{1}{4}, \frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2} + \text{BesselY}\left(\frac{1}{4}, \frac{1}{2}u^2\right)} \quad 0 < u \right\}^2$$

$$+ 2 \left(\right.$$

$$\begin{cases} \dfrac{\left(\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & u<0 \\ 1 & u=0 \\ \dfrac{\left(-\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{\dfrac{\left(-\Gamma\left(\frac{3}{4}\right)^2+\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & 0<u \end{cases}$$

$$\left(\right)$$

$$\begin{cases} \dfrac{\left(\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2+2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2+2\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & u<0 \\ 2 & u=0 \\ \dfrac{\left(\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2-2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2-2\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & 0<u \end{cases}$$

$$\begin{cases} \dfrac{\left(\dfrac{\left(-\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)-\frac{3}{2}\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)}{u^2}\right)u\left(\Gamma\left(\frac{3}{4}\right)^2+2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\left(-\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u\right.\right.\right.}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2+2\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right.} & \\ \dfrac{\left(\dfrac{\left(-\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)-\frac{3}{2}\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)}{u^2}\right)u\left(\Gamma\left(\frac{3}{4}\right)^2-2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\left(-\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u\right.\right.\right.}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2-2\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right.} & \end{cases}$$

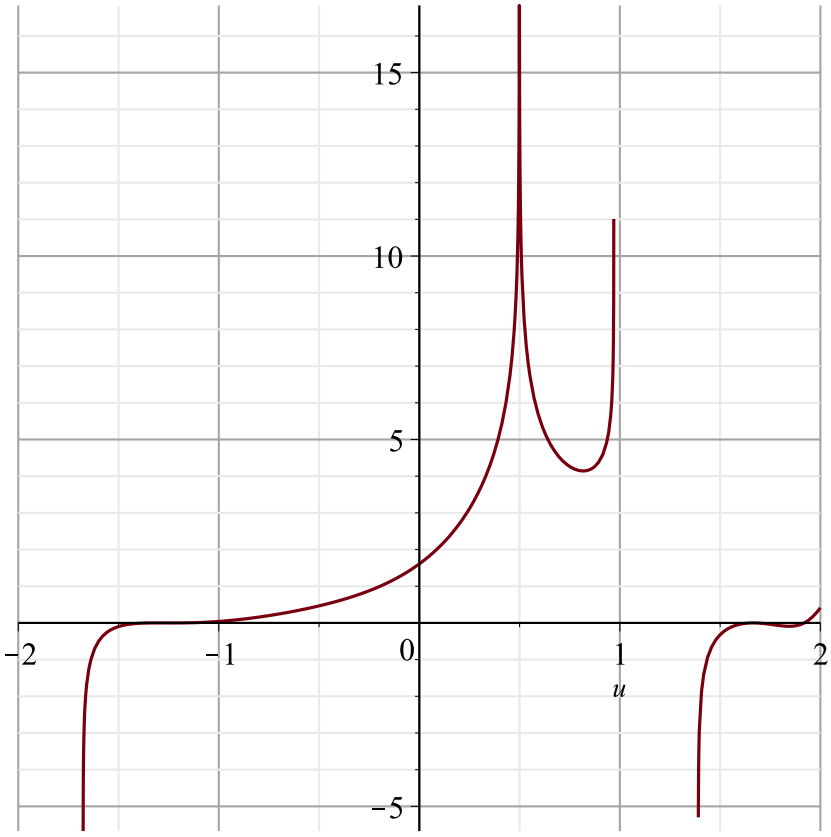
$$\left|\left|\left|

$$\left(\begin{array}{l}
\dfrac{\left(\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2+2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2+2\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & u<0 \\
2 & u=0 \\
\dfrac{\left(\dfrac{\text{BesselJ}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\left(\Gamma\left(\frac{3}{4}\right)^2-2\pi\right)}{\Gamma\left(\frac{3}{4}\right)^2}-\text{BesselY}\left(-\frac{3}{4},\frac{1}{2}u^2\right)\right)u}{-\dfrac{\left(\Gamma\left(\frac{3}{4}\right)^2-2\pi\right)\text{BesselJ}\left(\frac{1}{4},\frac{1}{2}u^2\right)}{\Gamma\left(\frac{3}{4}\right)^2}+\text{BesselY}\left(\frac{1}{4},\frac{1}{2}u^2\right)} & 0<u
\end{array}\right)$$

2

Graf over $f2'$:

```
> plot(f2(u), u = -2..2, gridlines)
```



> $f2'(0)$

0 **(9)**

NB: Selv om plottet viser, at $f2'$ er en differentiabel funktion i $u = 0$, så **svigter** kommandoen $f2'(0)$, idet funktionen er givet stykkevis: $u < 0$, $u = 0$ og $u > 0$.

Så må man tage grænseværdien i stedet for!

> $\lim_{u \to 0+} f2'(u)$

4 **(10)**

> $\lim_{u \to 0-} f2'(u)$

4 **(11)**

> $\lim_{u \to 0} f2'(u)$

4 **(12)**

Eller bruge **definitionen af differentialkvotient**:

> $\lim_{h \to 0} \frac{f2(0+h) - f2(0)}{h}$

4 **(13)**

Konklusion: $f2'(0) = 4$