Fladeintegral (2D-flade i 3 dimensioner)

restart
with(*plots*) :
with(*Integrator8*) :
with(*VektorAnalyse3*) = [*div*, *grad*, *kryds*, *prik*, *rot*, *vop*]

Parameterfremstilling af en flade i rummet (eksempel)

Parameterfremstilling af et 2D-område i planen, hvor $u \in \left[0;\ \frac{\pi}{2}\right]$ og $v \in [0;\ 1]$:

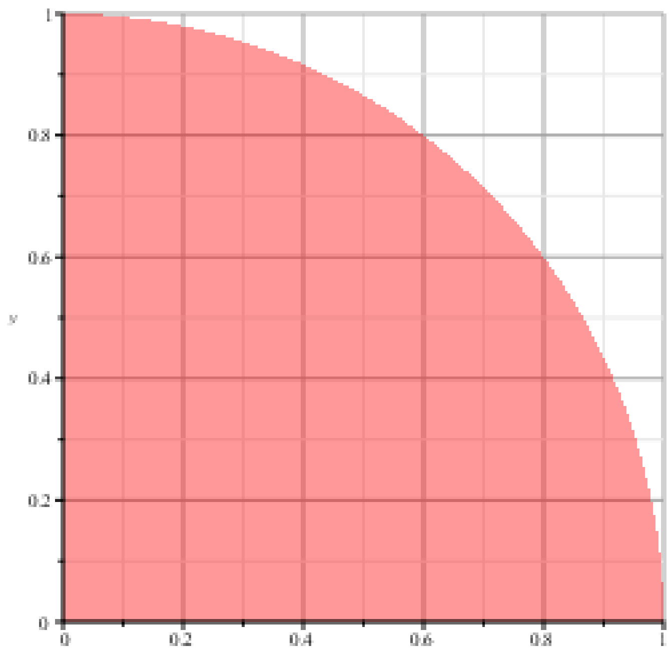
$r2(u, v) := \langle v\cdot\cos(u), v\cdot\sin(u)\rangle$:

$$r2(u, v) = \begin{bmatrix} v\cos(u) \\ v\sin(u) \end{bmatrix}$$

$INT := \left[0, \frac{\pi}{2}, 0, 1\right]$:

Graf over 2D-området i planen

2D-området i planen udgør en kvartcirkel med centrum i origo:
display(*plot2D*(*r2*(*u*, *v*), *INT*), *color* = *red*, *gridlines*, *style* = *surface*, *transparency* = 0.6, *view* = [0 ..1, 0 ..1], *labels* = [*x*, *y*])



2D-området i planen løftes op på grafen for *g* :

$g(x, y) := 1 - x\cdot y$:

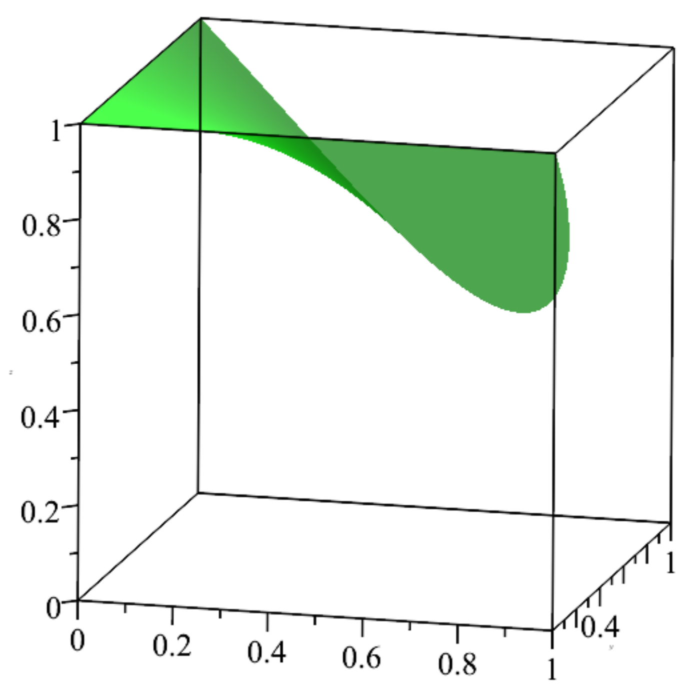
Graf over fladen i rummet

x og y koordinaten fra parametriseringen $r2$ genbruges, og z koordinaten findes ved at sammensætte g med $r2$:

$r(u, v) := \langle vop(r2(u, v)), g(vop(r2(u, v)))\rangle :$

$$r(u, v) = \begin{bmatrix} v\cos(u) \\ v\sin(u) \\ 1 - v^2 \sin(u)\cos(u) \end{bmatrix}$$

$G := plot3d\left(r(u, v), u = 0 .. \frac{\pi}{2}, v = 0 ..1, color = green, style = surface, transparency = 0.3, view = [0 ..1, 0 ..1, 0 ..1], labels = [x, y, z]\right)$



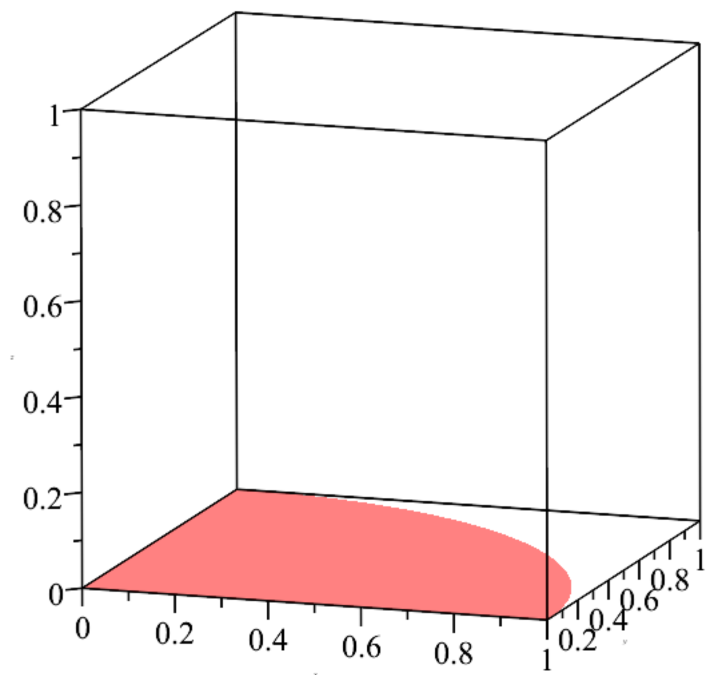
Området på gulvet ønskes medtegnet:

$r3(u, v) := \langle vop(r2(u, v)), 0\rangle :$

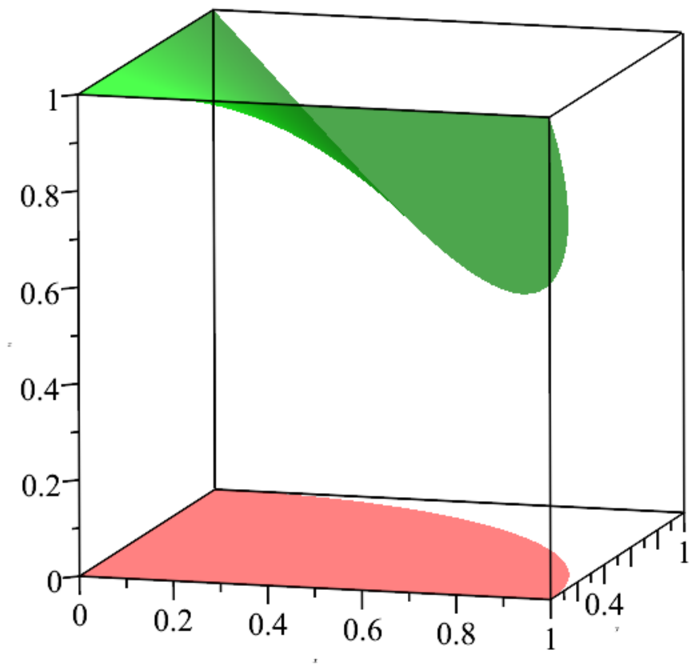
$$r3(u, v) = \begin{bmatrix} v\cos(u) \\ v\sin(u) \\ 0 \end{bmatrix}$$

$G3 := plot3d\left(r3(u, v), u = 0 .. \frac{\pi}{2}, v = 0 ..1, color = red, style = surface, transparency = 0.5, view\right.$

$= [0..1, 0..1, 0..1], labels = [x, y, z]\Big)$



$display(G, G3)$



Areal af fladen

Trinvis

Først beregnes de partielle afledede.

$$rm1 := diff(r(u,v),u) = \begin{bmatrix} -v\sin(u) \\ v\cos(u) \\ -v^2\cos(u)^2 + v^2\sin(u)^2 \end{bmatrix}$$

$$rm2 := diff(r(u,v),v) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ -2\,v\sin(u)\cos(u) \end{bmatrix}$$

Krydsproduktet beregnes:

$$N := kryds(rm1, rm2) = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

eller

$$N := rm1 \times rm2 = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

Jacobi-funktionen beregnes så som længden af normalvektoren N:

$Jacobi := LinearAlgebra[Norm](N, 2) =$

$$\left(\left|2\,v^2\cos(u)^2\sin(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right|^2 + \left|2\,v^2\sin(u)^2\cos(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right|^2 + \left|v\sin(u)^2 + v\cos(u)^2\right|^2\right)^{1/2}$$

eller

$Jacobi := \sqrt{prik(N, N)} =$

$$\left(\left(-2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right)^2 + \left(-2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right)^2 + \left(-v\sin(u)^2 - v\cos(u)^2\right)^2\right)^{1/2}$$

$Jacobi := simplify(\%) = \sqrt{v^2\,(v^2+1)}$

$Jacobi := simplify(\%)$ assuming $v > 0 = v\sqrt{v^2+1}$

Arealet bestemmes så ved et dobbeltintegral:

$$\int_0^1\int_0^{\frac{\pi}{2}} 1\cdot Jacobi\, du\, dv = \frac{\pi\left(-\frac{1}{3} + \frac{2\sqrt{2}}{3}\right)}{2}$$

Med færdig formel

$$\int_0^1\int_0^{\frac{\pi}{2}} 1\cdot LinearAlgebra[Norm](kryds(diff(r(u,v),u), diff(r(u,v),v)), 2)\, du\, dv = \frac{\pi\left(-\frac{1}{3} + \frac{2\sqrt{2}}{3}\right)}{2}$$

eller

$$\int_0^1\int_0^{\frac{\pi}{2}} 1\cdot\sqrt{prik(kryds(diff(r(u,v),u), diff(r(u,v),v)), kryds(diff(r(u,v),u), diff(r(u,v),v)))}\, du\, dv = \frac{\pi\left(-\frac{1}{3} + \frac{2\sqrt{2}}{3}\right)}{2}$$

Med Integrator8-pakken

$$INT = \left[0, \frac{\pi}{2}, 0, 1\right]$$

$$fladeIntGo(r, INT, 1) = -\frac{\pi}{6} + \frac{\sqrt{2}\,\pi}{3}$$

$$evalf(\%) = 0.9573622032$$

Konklusion: arealet af fladen er ca. 0.96

Masse af fladen

Hvis man antager, at fladen har en massetæthed (masse pr. arealenhed) kaldet f, så kan man beregne fladens samlede masse.

I dette eksempel afhænger massetætheden alene af z:
$f(x, y, z) := z:$

Trinvis

Først beregnes de partielle afledede.

$$rm1 := diff(r(u, v), u) = \begin{bmatrix} -v\sin(u) \\ v\cos(u) \\ -v^2\cos(u)^2 + v^2\sin(u)^2 \end{bmatrix}$$

$$rm2 := diff(r(u, v), v) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ -2\,v\sin(u)\cos(u) \end{bmatrix}$$

Krydsproduktet beregnes:

$$N := kryds(rm1, rm2) = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

eller

$$N := rm1 \times rm2 = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

Jacobi-funktionen beregnes så som længden af normalvektoren N:
$Jacobi := LinearAlgebra[Norm](N, 2) =$

$$\left(\left|2\,v^2\cos(u)^2\sin(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right|^2 + \left|2\,v^2\sin(u)^2\cos(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right|^2 + \left|v\sin(u)^2 + v\cos(u)^2\right|^2\right)^{1/2}$$

eller

$Jacobi := \sqrt{prik(N, N)} =$

$$\left(\left(-2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right)^2 + \left(-2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right)^2 + \left(-v\sin(u)^2 - v\cos(u)^2\right)^2\right)^{1/2}$$

$Jacobi := simplify(\%) = \sqrt{v^2\,(v^2+1)}$

$Jacobi := simplify(\%)$ assuming $v > 0 = v\sqrt{v^2+1}$

$$\int_0^1 \int_0^{\frac{\pi}{2}} f(vop(r(u,v)))\cdot Jacobi\, \mathrm{d}u\, \mathrm{d}v = -\frac{1}{15} - \frac{\pi}{6} + \frac{\sqrt{2}\,\pi}{3} - \frac{\sqrt{2}}{15}$$

Med færdig formel

$$\int_0^1 \int_0^{\frac{\pi}{2}} f(vop(r(u,v)))\cdot LinearAlgebra[Norm](kryds(diff(r(u,v),u), diff(r(u,v),v)), 2)\, \mathrm{d}u\, \mathrm{d}v$$

$$= -\frac{1}{15} - \frac{\pi}{6} + \frac{\sqrt{2}\,\pi}{3} - \frac{\sqrt{2}}{15}$$

eller

$$\int_0^1 \int_0^{\frac{\pi}{2}} f(vop(r(u,v))) \cdot\sqrt{prik(kryds(diff(r(u,v),u), diff(r(u,v),v)), kryds(diff(r(u,v),u), diff(r(u,v),v)))}\, \mathrm{d}u\, \mathrm{d}v$$

$$= -\frac{1}{15} - \frac{\pi}{6} + \frac{\sqrt{2}\,\pi}{3} - \frac{\sqrt{2}}{15}$$

Med Integrator8-pakken

$INT = \left[0, \frac{\pi}{2}, 0, 1\right]$

$$fladeIntGo(r, INT, f) = \frac{(10\,\pi - 2)\sqrt{2}}{30} - \frac{\pi}{6} - \frac{1}{15}$$

$evalf(\%) = 0.7964146325$

Konklusion: fladens masse er ca. $\underline{\underline{0.80}}$