Fladeintegral (2D-flade i 3 dimensioner)

restart
with(*plots*) :
with(*Integrator8*) :
with(*VektorAnalyse4*) = [*div*, *grad*, *kryds*, *prik*, *rot*, *vop*]
with(*plot2D3D2*) = [*NormalVektorer*, *TangentVektorer*, *plot2D*, *plot3D*]

Parameterfremstilling af en flade i rummet (eksempel)

Parameterfremstilling af et 2D-område i planen, hvor $u \in \left[0;\, \frac{\pi}{2}\right]$ og $v \in [0;\,1]$:

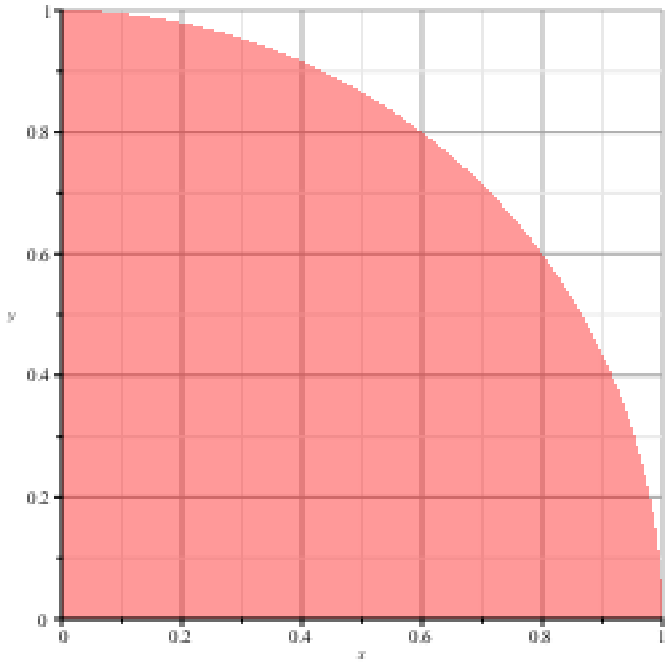
$r2(u, v) := \langle v \cdot \cos(u), v \cdot \sin(u) \rangle$:

$$r2(u, v) = \begin{bmatrix} v\cos(u) \\ v\sin(u) \end{bmatrix}$$

$INT := \left[0, \frac{\pi}{2}, 0, 1\right]$:

Graf over 2D-området i planen

2D-området i planen udgør en kvartcirkel med centrum i origo:
display(*plot2D*(*r2*(*u*, *v*), *INT*), *color* = *red*, *gridlines*, *style* = *surface*, *transparency* = 0.6, *view* = [0 ..1, 0 ..1], *labels* = [*x*, *y*])



2D-området i planen løftes op på grafen for *g* :

$g(x, y) := 1 - x \cdot y$:

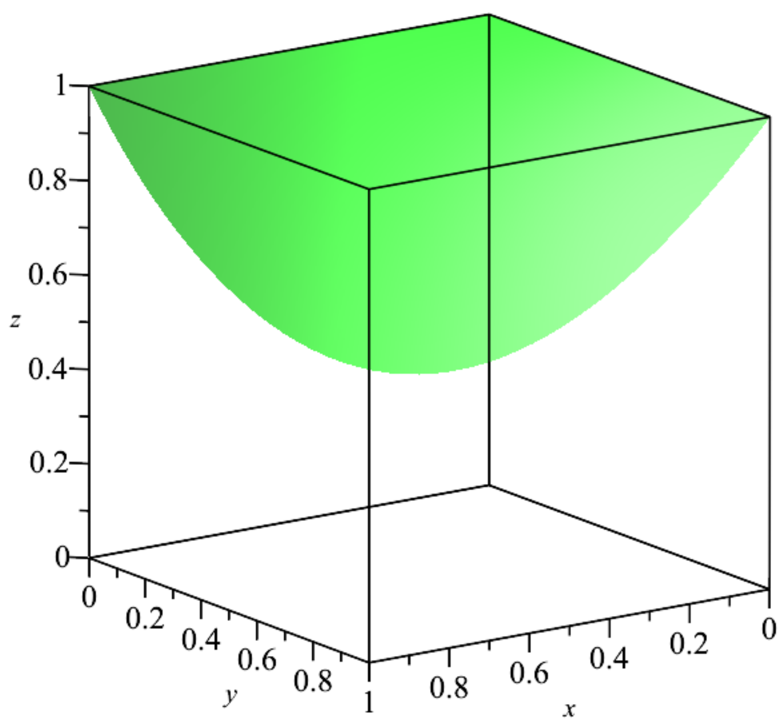
Graf over fladen i rummet

x og *y* koordinaten fra parametriseringen *r2* genbruges, og *z* koordinaten findes ved at sammensætte *g* med *r2* :
$r(u, v) := \langle vop(r2(u, v)), g(vop(r2(u, v))) \rangle$:

$r(u, v) =$

$$\begin{bmatrix} v\cos(u) \\ v\sin(u) \\ 1 - v^2\sin(u)\cos(u) \end{bmatrix}$$

$G := plot3d\left(r(u, v), u = 0..\frac{\pi}{2}, v = 0..1, color = green, style = surface, transparency = 0.3, view = [0..1, 0..1, 0..1], labels = [x, y, z]\right)$

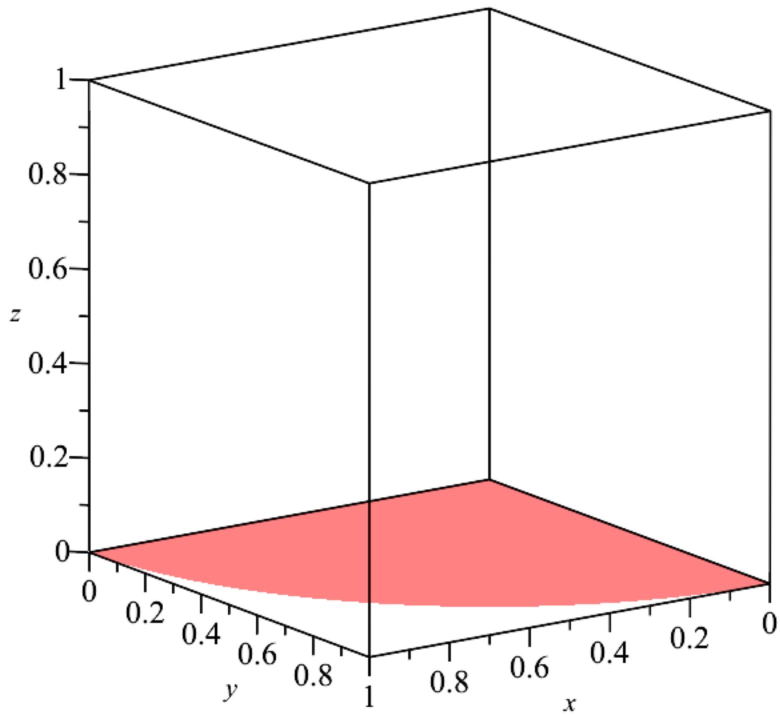


Området på gulvet ønskes medtegnet:

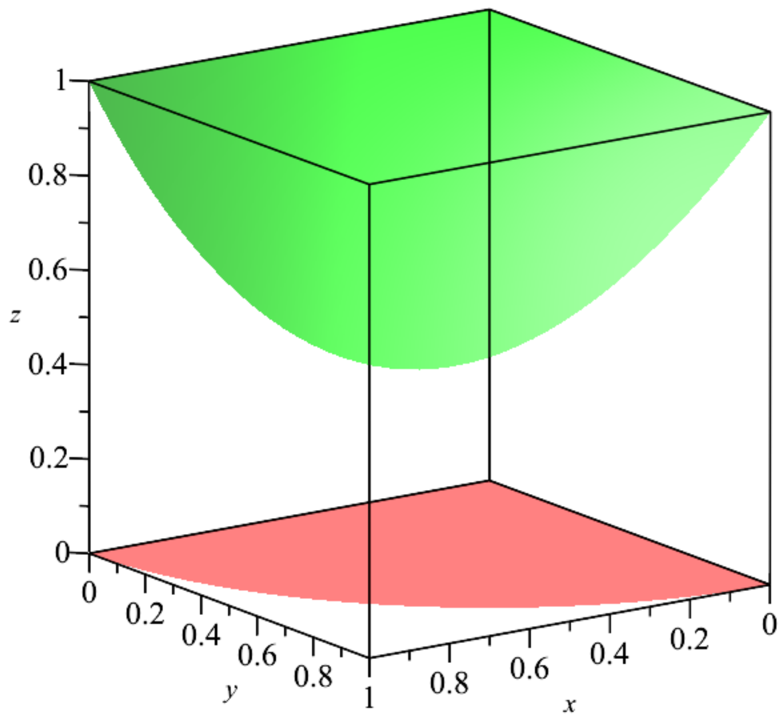
$r3(u, v) := \langle vop(r2(u, v)), 0\rangle$:

$$r3(u, v) = \begin{bmatrix} v\cos(u) \\ v\sin(u) \\ 0 \end{bmatrix}$$

$G3 := plot3d\left(r3(u, v), u = 0..\frac{\pi}{2}, v = 0..1, color = red, style = surface, transparency = 0.5, view = [0..1, 0..1, 0..1], labels = [x, y, z]\right)$



display(*G*, *G3*)



Areal af fladen

Trinvis

Først beregnes de partielle afledede.

$$rm1 := diff(r(u,v),u) = \begin{bmatrix} -v\sin(u) \\ v\cos(u) \\ -v^2\cos(u)^2 + v^2\sin(u)^2 \end{bmatrix}$$

$$rm2 := diff(r(u,v),v) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ -2\,v\sin(u)\cos(u) \end{bmatrix}$$

Krydsproduktet beregnes:

$$N := kryds(rm1, rm2) = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

eller

$$N := rm1 \times rm2 = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

Jacobi-funktionen beregnes så som længden af normalvektoren N:

$$Jacobi := LinearAlgebra[Norm](N, 2) =$$

$$\left(\left|2\,v^2\cos(u)^2\sin(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right|^2 + \left|2\,v^2\sin(u)^2\cos(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right|^2 + \left|v\sin(u)^2 + v\cos(u)^2\right|^2\right)^{1/2}$$

eller

$$Jacobi := \sqrt{prik(N, N)} =$$

$$\left(\left(-2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right)^2 + \left(-2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right)^2 + \left(-v\sin(u)^2 - v\cos(u)^2\right)^2\right)^{1/2}$$

$$Jacobi := simplify(\%) = \sqrt{v^2\left(v^2+1\right)}$$

$$Jacobi := simplify(\%) \text{ assuming } v > 0 = v\sqrt{v^2+1}$$

Arealet bestemmes så ved et dobbeltintegral:

$$\int_0^1 \int_0^{\frac{\pi}{2}} 1 \cdot Jacobi \, du \, dv = \frac{\pi\left(-\frac{1}{3} + \frac{2\sqrt{2}}{3}\right)}{2}$$

Med færdig formel

$$\int_0^1 \int_0^{\frac{\pi}{2}} 1 \cdot LinearAlgebra[Norm](kryds(diff(r(u,v),u), diff(r(u,v),v)), 2) \, du \, dv = \frac{\pi\left(-\frac{1}{3} + \frac{2\sqrt{2}}{3}\right)}{2}$$

$$simplify(\%) = -\frac{\pi}{6} + \frac{\sqrt{2}\,\pi}{3}$$

eller

$$\int_0^1 \int_0^{\frac{\pi}{2}} 1 \cdot \sqrt{prik(kryds(diff(r(u,v),u), diff(r(u,v),v)), kryds(diff(r(u,v),u), diff(r(u,v),v)))} \, du \, dv =$$

$$\frac{\pi\left(-\frac{1}{3}+\frac{2\sqrt{2}}{3}\right)}{2}$$

$$simplify(\%) = -\frac{\pi}{6}+\frac{\sqrt{2}\,\pi}{3}$$

Med Integrator8-pakken

$$INT = \left[0, \frac{\pi}{2}, 0, 1\right]$$

$$fladeIntGo(r, INT, 1) = -\frac{\pi}{6}+\frac{\sqrt{2}\,\pi}{3}$$

$$evalf(\%) = 0.9573622032$$

Konklusion: arealet af fladen er ca. 0.96

Masse af fladen

Hvis man antager, at fladen har en massetæthed (masse pr. arealenhed) kaldet f, så kan man beregne fladens samlede masse.

I dette eksempel afhænger massetætheden alene af z:
$f(x, y, z) := z :$

Trinvis

Først beregnes de partielle afledede.

$$rm1 := diff(r(u,v), u) = \begin{bmatrix} -v\sin(u) \\ v\cos(u) \\ -v^2\cos(u)^2 + v^2\sin(u)^2 \end{bmatrix}$$

$$rm2 := diff(r(u,v), v) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ -2\,v\sin(u)\cos(u) \end{bmatrix}$$

Krydsproduktet beregnes:

$$N := kryds(rm1, rm2) = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

eller

$$N := rm1 \times rm2 = \begin{bmatrix} -2\,v^2\cos(u)^2\sin(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u) \\ -2\,v^2\sin(u)^2\cos(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u) \\ -v\sin(u)^2 - v\cos(u)^2 \end{bmatrix}$$

Jacobi-funktionen beregnes så som længden af normalvektoren N:
$Jacobi := LinearAlgebra[Norm](N, 2) =$

$$\left(\left|2\,v^2\cos(u)^2\sin(u) + \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\sin(u)\right|^2 + \left|2\,v^2\sin(u)^2\cos(u) - \left(-v^2\cos(u)^2 + v^2\sin(u)^2\right)\cos(u)\right|^2 + \left|v\sin(u)^2 + v\cos(u)^2\right|^2\right)^{1/2}$$

eller

$Jacobi := \sqrt{prik(N, N)} =$

$$\left(\left(-2\,v^2\cos(u)^2\sin(u)-\left(-v^2\cos(u)^2+v^2\sin(u)^2\right)\sin(u)\right)^2+\left(-2\,v^2\sin(u)^2\cos(u)+\left(-v^2\cos(u)^2+v^2\sin(u)^2\right)\cos(u)\right)^2+\left(-v\sin(u)^2-v\cos(u)^2\right)^2\right)^{1/2}$$

$$Jacobi := simplify(\%) = \sqrt{v^2\left(v^2+1\right)}$$

$$Jacobi := simplify(\%) \text{ assuming } v > 0 = v\sqrt{v^2+1}$$

$$\int_0^1\int_0^{\frac{\pi}{2}} f(vop(r(u,v)))\cdot Jacobi\,\mathrm{d}u\,\mathrm{d}v = -\frac{1}{15}-\frac{\pi}{6}+\frac{\sqrt{2}\,\pi}{3}-\frac{\sqrt{2}}{15}$$

$$simplify(\%) = \frac{(10\,\pi-2)\sqrt{2}}{30}-\frac{\pi}{6}-\frac{1}{15}$$

Med færdig formel

$$\int_0^1\int_0^{\frac{\pi}{2}} f(vop(r(u,v)))\cdot LinearAlgebra[Norm](kryds(diff(r(u,v),u),diff(r(u,v),v)),2)\,\mathrm{d}u\,\mathrm{d}v =$$

$$-\frac{1}{15}-\frac{\pi}{6}+\frac{\sqrt{2}\,\pi}{3}-\frac{\sqrt{2}}{15}$$

$$simplify(\%) = \frac{(10\,\pi-2)\sqrt{2}}{30}-\frac{\pi}{6}-\frac{1}{15}$$

eller

$$\int_0^1\int_0^{\frac{\pi}{2}} f(vop(r(u,v))) \cdot\sqrt{prik(kryds(diff(r(u,v),u),diff(r(u,v),v)),kryds(diff(r(u,v),u),diff(r(u,v),v)))}\,\mathrm{d}u\,\mathrm{d}v$$

$$= -\frac{1}{15}-\frac{\pi}{6}+\frac{\sqrt{2}\,\pi}{3}-\frac{\sqrt{2}}{15}$$

$$simplify(\%) = \frac{(10\,\pi-2)\sqrt{2}}{30}-\frac{\pi}{6}-\frac{1}{15}$$

Med Integrator8-pakken

$$INT = \left[0, \frac{\pi}{2}, 0, 1\right]$$

$$fladeIntGo(r, INT, f) = \frac{(10\,\pi-2)\sqrt{2}}{30}-\frac{\pi}{6}-\frac{1}{15}$$

$$evalf(\%) = 0.7964146325$$

Konklusion: fladens masse er ca. 0.80