Analytisk funktion

En C^{∞} – funktion, hvis Taylor-række konvergerer imod funktionen i en omegn af punktet x_0, siges at være en **analytisk funktion**.
Læs nærmere: **http://en.wikipedia.org/wiki/Analytic_function**

Ikke alle funktion, som kan differentieres uendelig mange gange, er analytiske!

Modeksempel:

Definer følgende funktion:

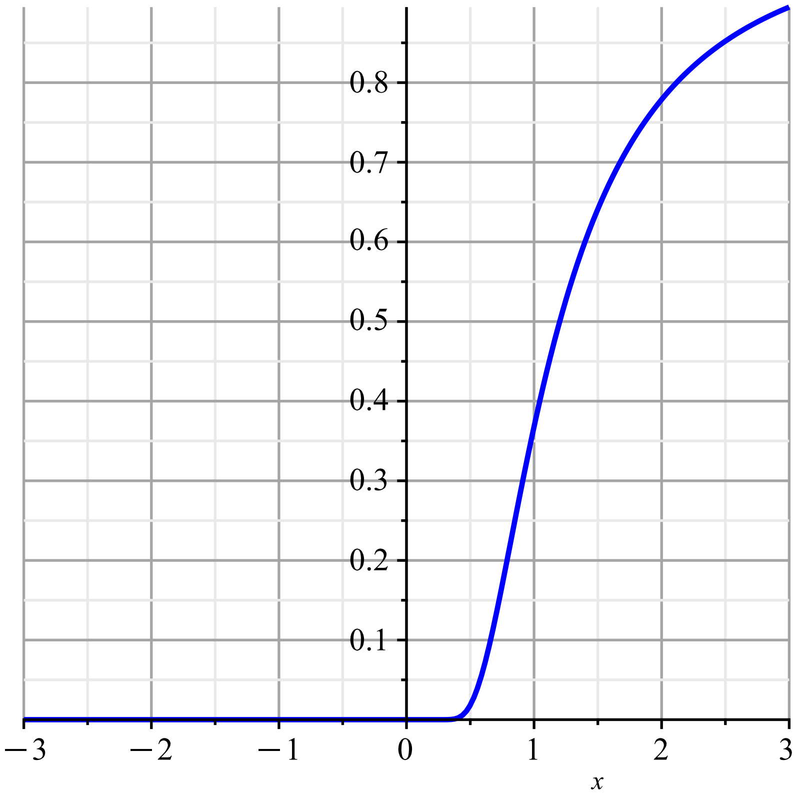
$$f(x) = \begin{cases} 0 & \text{for } x \le 0 \\ e^{-\frac{1}{x^2}} & \text{for } x > 0 \end{cases}$$

Denne funktion $f \in C^{\infty}$, men er IKKE en analytisk funktion, da Taylor-rækken ud fra ud fra $x_0 = 0$ er identisk med 0 !

> *restart*

> $f(x) := \begin{cases} 0 & x \le 0 \\ e^{-\frac{1}{x^2}} & x > 0 \end{cases}$:

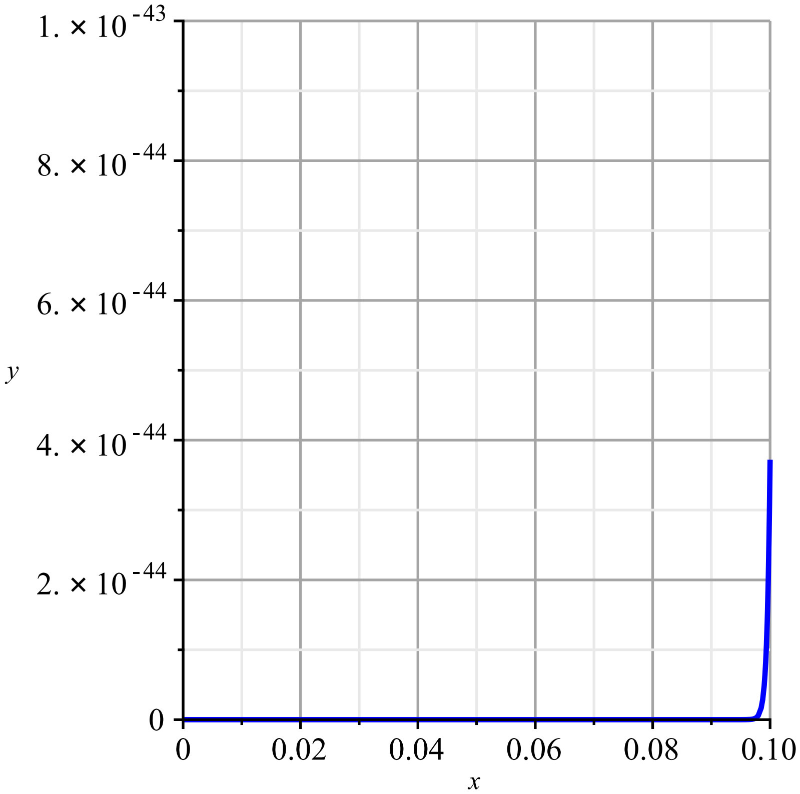
> $plot(f(x), x = -3..3, color = blue, thickness = 2, gridlines)$



> $f(0.1)$

$$3.720075976 \times 10^{-44} \quad (1)$$

> $plot\left(f(x), x = 0..0.1, y = 0..10^{-43}, color = blue, thickness = 2, gridlines\right)$



Funktionen er så flad i $x_0 = 0$, at alle afledede i dette punkt er 0!

Derfor er Taylorrækken ud fra $x_0 = 0$ identisk med nulpolynomiet.
Det betyder, at Taylorrækken ud fra $x_0 = 0$ ALDRIG konvergerer imod $f(x)$ for noget $x > 0$.

> $\lim_{x \to 0+} (f(x))$

$$0 \tag{2}$$

Dvs. $f(0) = 0$, og f er kontinuert i 0.

> $f_1 := unapply\left(\frac{d}{dx} f(x), x\right)$

$$f_1 := x \mapsto \begin{cases} 0 & x \le 0 \\ \frac{2 \cdot e^{-\frac{1}{x^2}}}{x^3} & 0 < x \end{cases} \tag{3}$$

> $\lim_{x \to 0+} (f_1(x))$

$$0 \tag{4}$$

Dvs. $f\,'(0) = 0$, og $f\,'$ er kontinuert i 0.

> $f_2 := unapply\left(\frac{d}{dx} f_1(x), x\right)$

$$f_2 := x \mapsto \begin{cases} 0 & x \le 0 \\ -\frac{6 \cdot e^{-\frac{1}{x^2}}}{x^4} + \frac{4 \cdot e^{-\frac{1}{x^2}}}{x^6} & 0 < x \end{cases} \tag{5}$$

> $\lim_{x \to 0+} (f_2(x))$

$$0 \quad \mathbf{(6)}$$

> $\lim_{x \to 0+} \left(\frac{f_1(x) - 0}{x - 0} \right)$

$$0 \quad \mathbf{(7)}$$

Dvs. $f\,''(0) = 0$, og $f\,''$ er kontinuert i 0.

osv. osv.

> **for** n **from** 2 **by** 1 **to** 10 **do** $f_{n+1} := unapply\left(\frac{d}{dx} f_n(x), x\right)$ **end do**

$$f_3 := x \mapsto \begin{cases} 0 & x \le 0 \\ \frac{24 \cdot e^{-\frac{1}{x^2}}}{x^5} - \frac{36 \cdot e^{-\frac{1}{x^2}}}{x^7} + \frac{8 \cdot e^{-\frac{1}{x^2}}}{x^9} & 0 < x \end{cases}$$

$$f_4 := x \mapsto \begin{cases} 0 & x \le 0 \\ -\frac{120 \cdot e^{-\frac{1}{x^2}}}{x^6} + \frac{300 \cdot e^{-\frac{1}{x^2}}}{x^8} - \frac{144 \cdot e^{-\frac{1}{x^2}}}{x^{10}} + \frac{16 \cdot e^{-\frac{1}{x^2}}}{x^{12}} & 0 < x \end{cases}$$

$$f_5 := x \mapsto \begin{cases} 0 & x \le 0 \\ \frac{720 \cdot e^{-\frac{1}{x^2}}}{x^7} - \frac{2640 \cdot e^{-\frac{1}{x^2}}}{x^9} + \frac{2040 \cdot e^{-\frac{1}{x^2}}}{x^{11}} - \frac{480 \cdot e^{-\frac{1}{x^2}}}{x^{13}} + \frac{32 \cdot e^{-\frac{1}{x^2}}}{x^{15}} & 0 < x \end{cases}$$

$$f_6 := x \mapsto \begin{cases} 0 & x \le \\ -\frac{5040 \cdot e^{-\frac{1}{x^2}}}{x^8} + \frac{25200 \cdot e^{-\frac{1}{x^2}}}{x^{10}} - \frac{27720 \cdot e^{-\frac{1}{x^2}}}{x^{12}} + \frac{10320 \cdot e^{-\frac{1}{x^2}}}{x^{14}} - \frac{1440 \cdot e^{-\frac{1}{x^2}}}{x^{16}} + \frac{64 \cdot e^{-\frac{1}{x^2}}}{x^{18}} & 0 < \end{cases}$$

$$f_7 := x \mapsto \begin{cases} 0 \\ \frac{40320 \cdot e^{-\frac{1}{x^2}}}{x^9} - \frac{262080 \cdot e^{-\frac{1}{x^2}}}{x^{11}} + \frac{383040 \cdot e^{-\frac{1}{x^2}}}{x^{13}} - \frac{199920 \cdot e^{-\frac{1}{x^2}}}{x^{15}} + \frac{43680 \cdot e^{-\frac{1}{x^2}}}{x^{17}} - \frac{4032 \cdot e^{-\frac{1}{x^2}}}{x^{19}} \end{cases}$$

$$f_8 := x \mapsto \begin{cases} 0 \\ -\frac{362880 \cdot e^{-\frac{1}{x^2}}}{x^{10}} + \frac{2963520 \cdot e^{-\frac{1}{x^2}}}{x^{12}} - \frac{5503680 \cdot e^{-\frac{1}{x^2}}}{x^{14}} + \frac{3764880 \cdot e^{-\frac{1}{x^2}}}{x^{16}} - \frac{1142400 \cdot e^{-\frac{1}{x^2}}}{x^{18}} + 16 \end{cases}$$

$$f_9 := x \mapsto \begin{cases} 0 \\ \frac{3628800 \cdot e^{-\frac{1}{x^2}}}{x^{11}} - \frac{36288000 \cdot e^{-\frac{1}{x^2}}}{x^{13}} + \frac{82978560 \cdot e^{-\frac{1}{x^2}}}{x^{15}} - \frac{71245440 \cdot e^{-\frac{1}{x^2}}}{x^{17}} + \frac{28092960 \cdot e^{-\frac{1}{x^2}}}{x^{19}} - \end{cases}$$

$$f_{10} := x \mapsto \begin{cases} -\frac{39916800 \cdot e^{-\frac{1}{x^2}}}{x^{12}} + \frac{479001600 \cdot e^{-\frac{1}{x^2}}}{x^{14}} - \frac{1317254400 \cdot e^{-\frac{1}{x^2}}}{x^{16}} + \frac{1377129600 \cdot e^{-\frac{1}{x^2}}}{x^{18}} - \frac{67625712}{x^2} \end{cases}$$

$$f_{11} := x \mapsto \begin{cases} \frac{479001600 \cdot e^{-\frac{1}{x^2}}}{x^{13}} - \frac{6785856000 \cdot e^{-\frac{1}{x^2}}}{x^{15}} + \frac{22034073600 \cdot e^{-\frac{1}{x^2}}}{x^{17}} - \frac{27422841600 \cdot e^{-\frac{1}{x^2}}}{x^{19}} + 162794 \end{cases}$$

> **for** n **from** 2 **by** 1 **to** 10 **do** $\lim_{x \to 0+} \left(f_{n+1}(x)\right)$ **end do**

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$

$$0$$
$$0 \tag{9}$$

> **for** n **from** 2 **by** 1 **to** 10 **do** $\lim_{x \to 0+} \left(\frac{f_n(x) - 0}{x - 0} \right)$ **end do**

$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0$$
$$0 \tag{10}$$