Effektiv løsning af differentialligningssystem med "dsolve"; hvor problemet opskrives på matrix-form

Opgave

Løs differentialigningssystemet:

$$\begin{bmatrix} x_1'(t) \\ x_2'(t) \\ x_3'(t) \end{bmatrix} = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix} \cdot \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix} + \begin{bmatrix} 2\cdot e^t \\ \sin(t) \\ t^2 + 2\cdot t \end{bmatrix}$$

Fuldstændig løsning

restart

with(*LinearAlgebra*) :

$X := \langle x_1(t), x_2(t), x_3(t) \rangle$:

$$X = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$$

$Xm := diff(X, t)$:

$$Xm = \begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \\ \frac{d}{dt} x_3(t) \end{bmatrix}$$

$A := \langle 1, -1, 1; 2, 4, -1; 0, 0, 3 \rangle$:

$$A = \begin{bmatrix} 1 & -1 & 1 \\ 2 & 4 & -1 \\ 0 & 0 & 3 \end{bmatrix}$$

$B := \langle 2\cdot e^t, \sin(t), t^2 + 2\cdot t \rangle$:

$$B = \begin{bmatrix} 2\, e^t \\ \sin(t) \\ t^2 + 2\, t \end{bmatrix}$$

Differentialligningssystemet lyder så:

$$Xm = A \cdot X + B = \begin{bmatrix} \frac{d}{dt} x_1(t) \\ \frac{d}{dt} x_2(t) \\ \frac{d}{dt} x_3(t) \end{bmatrix} = \begin{bmatrix} x_1(t) - x_2(t) + x_3(t) + 2\, e^t \\ 2\, x_1(t) + 4\, x_2(t) - x_3(t) + \sin(t) \\ 3\, x_3(t) + t^2 + 2\, t \end{bmatrix}$$

Systemet kan nu løses med *dsolve* på **matrixform**:

$L := \textbf{dsolve}(Xm = A \cdot X + B)$

$$L := \left\{ x_1(t) = e^{2t}\_C2 + e^{3t}\_C1 + \frac{t^2}{6} - \frac{\cos(t)}{10} - \frac{\sin(t)}{10} - 3\,e^t + \frac{11\,t}{18} + \frac{49}{108}, x_2(t) = -e^{2t}\_C2 - 2\,e^{3t}\_C1 - \frac{11\,t}{18} - \frac{\sin(t)}{5} + 2\,e^t - \frac{49}{108} - \frac{t^2}{6} + e^{3t}\_C3, x_3(t) = -\frac{t^2}{3} - \frac{8\,t}{9} - \frac{8}{27} + e^{3t}\_C3 \right\} \qquad (1.1)$$

$xLØS_1 := unapply(rhs(L[1]), t) : xLØS_2 := unapply(rhs(L[2]), t) : xLØS_3 := unapply(rhs(L[3]), t) :$

Den fuldstændige løsning:

$$\begin{bmatrix} xLØS_1(t) \\ xLØS_2(t) \\ xLØS_3(t) \end{bmatrix} = \begin{bmatrix} e^{2t}\_C2 + e^{3t}\_C1 + \frac{t^2}{6} - \frac{\cos(t)}{10} - \frac{\sin(t)}{10} - 3\,e^t + \frac{11\,t}{18} + \frac{49}{108} \\ -e^{2t}\_C2 - 2\,e^{3t}\_C1 - \frac{11\,t}{18} - \frac{\sin(t)}{5} + 2\,e^t - \frac{49}{108} - \frac{t^2}{6} + e^{3t}\_C3 \\ -\frac{t^2}{3} - \frac{8\,t}{9} - \frac{8}{27} + e^{3t}\_C3 \end{bmatrix}$$

hvor de konstanter $t, \_C1, \_C2, \_C3 \in \mathbb{R}$

Partikulær løsning

Bestem den løsning, som opfylder at $(x_1(0), x_2(0), x_3(0)) = (1, 2, 3)$

Til ovenstående skal blot tilføje denne **betingelse**:

$X_0 := subs(t = 0, X) = \langle 1, 2, 3 \rangle :$

$$X_0 = \begin{bmatrix} x_1(0) \\ x_2(0) \\ x_3(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 2 \\ 3 \end{bmatrix}$$

Systemet kan nu løses *dsolve* på **matrixform**:

$L := \textbf{dsolve}(\{Xm = A \cdot X + B, X_0\}) =$

$$\left\{ x_1(t) = \frac{89}{20}\,e^{2t} - \frac{217}{270}\,e^{3t} + \frac{1}{6}\,t^2 - \frac{1}{10}\cos(t) - \frac{1}{10}\sin(t) - 3\,e^t + \frac{11}{18}\,t + \frac{49}{108}, x_2(t) = -\frac{89}{20}\,e^{2t} + \frac{662}{135}\,e^{3t} - \frac{11}{18}\,t - \frac{1}{5}\sin(t) + 2\,e^t - \frac{49}{108} - \frac{1}{6}\,t^2, x_3(t) = -\frac{1}{3}\,t^2 - \frac{8}{9}\,t - \frac{8}{27} + \frac{89}{27}\,e^{3t} \right\}$$

$xPAR_1 := unapply(rhs(L[1]), t) : xPAR_2 := unapply(rhs(L[2]), t) : xPAR_3 := unapply(rhs(L[3]), t) :$

Den partikulære løsning:

$$\begin{bmatrix} xPAR_1(t) \\ xPAR_2(t) \\ xPAR_3(t) \end{bmatrix} = \begin{bmatrix} \frac{89\,e^{2t}}{20} - \frac{217\,e^{3t}}{270} + \frac{t^2}{6} - \frac{\cos(t)}{10} - \frac{\sin(t)}{10} - 3\,e^t + \frac{11\,t}{18} + \frac{49}{108} \\ -\frac{89\,e^{2t}}{20} + \frac{662\,e^{3t}}{135} - \frac{11\,t}{18} - \frac{\sin(t)}{5} + 2\,e^t - \frac{49}{108} - \frac{t^2}{6} \\ -\frac{t^2}{3} - \frac{8\,t}{9} - \frac{8}{27} + \frac{89\,e^{3t}}{27} \end{bmatrix}$$ hvor $t \in \mathbb{R}$