Kurveintegral (i 2 eller 3 dimensioner)

$restart$
$with(plots):$
$with(Integrator8):$
$with(VektorAnalyse4) = [div, grad, kryds, prik, rot, vop]$

$\mathbb{R}^2$ (parabelstykke)

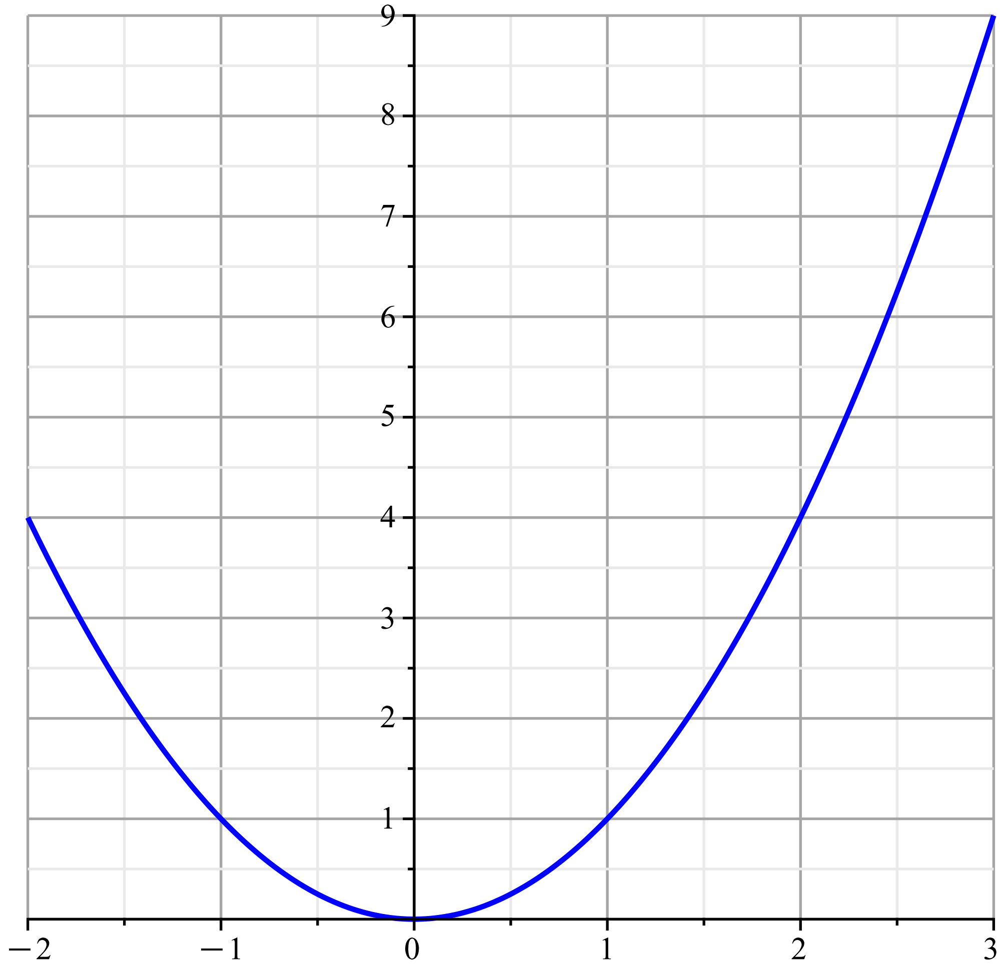
Parameterfremstilling af parablen, hvor $u \in [-2..3]$:

$r(u) := \langle u, u^2 \rangle :$

$$r(u) = \begin{bmatrix} u \\ u^2 \end{bmatrix}$$

Graf

$plot([vop(r(u)), u = -2..3], gridlines, color = blue, thickness = 2)$



Kurvelængde

Trinvis

$rm(u) := diff(r(u), u) :$

$$rm(u) = \begin{bmatrix} 1 \\ 2\,u \end{bmatrix}$$

$$Jacobi := \sqrt{prik(rm(u), rm(u))} = \sqrt{4\,u^2 + 1}$$

$$\int_{-2}^{3} 1 \cdot Jacobi \, du = \sqrt{17} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{3\sqrt{37}}{2} + \frac{\operatorname{arcsinh}(6)}{4}$$

Med færdig formel

$$\int_{-2}^{3} 1 \cdot LinearAlgebra[Norm](diff(r(u), u), 2) \, du = \sqrt{17} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{3\sqrt{37}}{2} + \frac{\operatorname{arcsinh}(6)}{4}$$

eller

$$\int_{-2}^{3} 1 \cdot \sqrt{prik(diff(r(u), u), diff(r(u), u))} \, du = \sqrt{17} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{3\sqrt{37}}{2} + \frac{\operatorname{arcsinh}(6)}{4}$$

Med Integrator8-pakken

$INT := [-2, 3]:$

$$kurveIntGo(r, INT, 1) = \sqrt{17} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{3\sqrt{37}}{2} + \frac{\operatorname{arcsinh}(6)}{4}$$

$evalf(\%) = 14.39387252$

Konklusion: kurvelængden er ca. 14.4

Masse af kurve

Hvis man antager, at kurven har en massetæthed (masse pr. længdeenhed) kaldet f, så kan man beregne kurvens samlede masse.

I dette eksempel afhænger massetætheden alene af x:
$f(x, y) := 2 \cdot x + 1:$

Trinvis

$rm(u) := diff(r(u), u):$

$$rm(u) = \begin{bmatrix} 1 \\ 2u \end{bmatrix}$$

$$Jacobi := \sqrt{prik(rm(u), rm(u))} = \sqrt{4u^2 + 1}$$

$$\int_{-2}^{3} f(vop(r(u))) \cdot Jacobi \, du = -\frac{11\sqrt{17}}{6} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{23\sqrt{37}}{3} + \frac{\operatorname{arcsinh}(6)}{4}$$

Med færdig formel

$$\int_{-2}^{3} f(vop(r(u))) \cdot LinearAlgebra[Norm](diff(r(u), u), 2) \, du =$$
$$-\frac{11\sqrt{17}}{6} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{23\sqrt{37}}{3} + \frac{\operatorname{arcsinh}(6)}{4}$$

eller

$$\int_{-2}^{3} f(vop(r(u))) \cdot \sqrt{prik(diff(r(u), u), diff(r(u), u))} \, du =$$
$$-\frac{11\sqrt{17}}{6} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{23\sqrt{37}}{3} + \frac{\operatorname{arcsinh}(6)}{4}$$

Med Integrator8-pakken

$INT := [-2, 3]:$

$$kurveIntGo(r, INT, f) = -\frac{11\sqrt{17}}{6} + \frac{\operatorname{arcsinh}(4)}{4} + \frac{23\sqrt{37}}{3} + \frac{\operatorname{arcsinh}(6)}{4}$$

$evalf(\%) = 40.22210885$

Konklusion: kurven masse er ca. 40.2

$\mathbb{R}^3$ (spiral, helix)

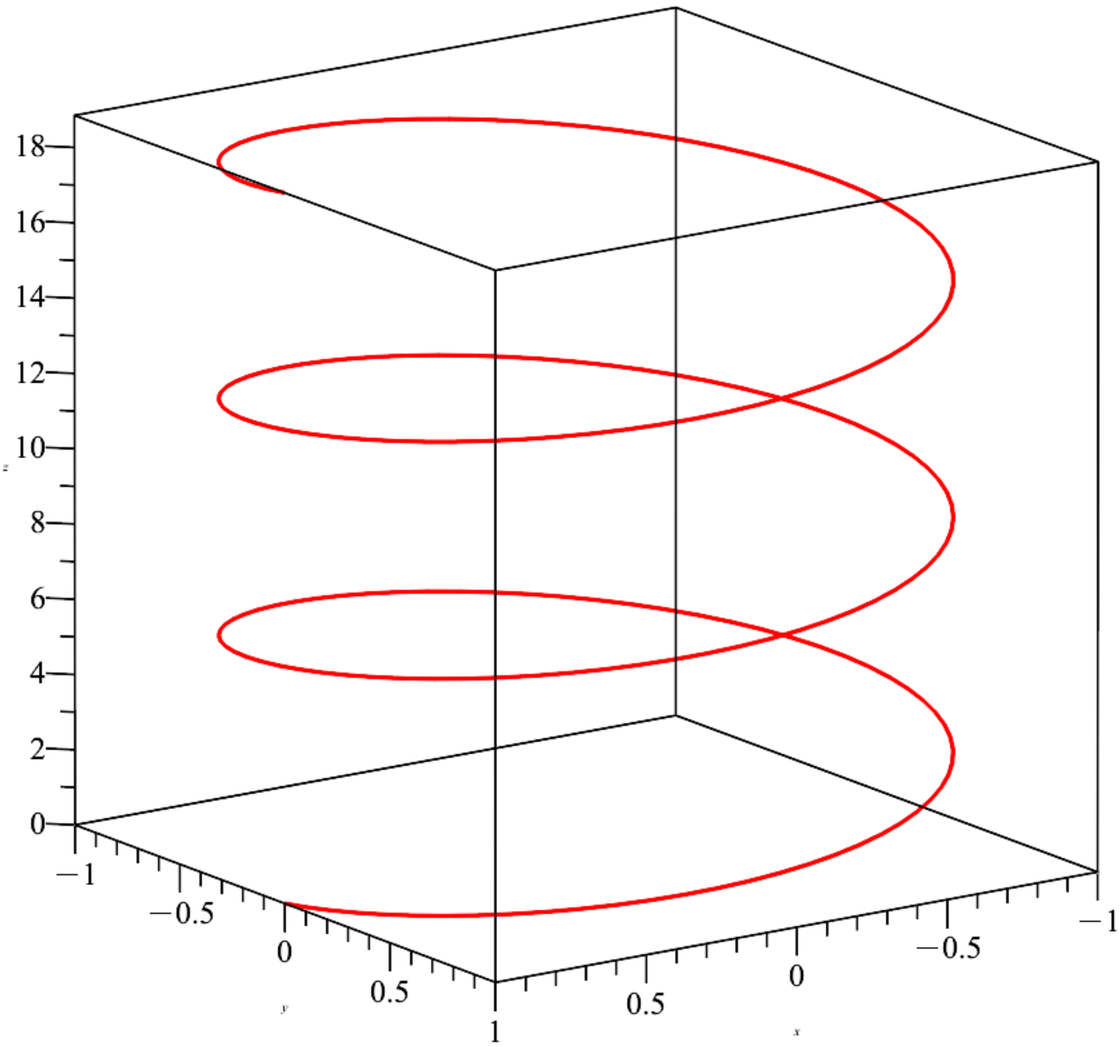
Parameterfremstilling af spiralen, hvor $u \in [0; 6\cdot\pi]$:

$r(u) := \langle \cos(u), \sin(u), u \rangle :$

$$r(u) = \begin{bmatrix} \cos(u) \\ \sin(u) \\ u \end{bmatrix}$$

Graf

$spacecurve([vop(r(u)), u = 0..6\cdot\pi], color = red, thickness = 3, labels = [x, y, z])$



Kurvelængde

Trinvis

$rm(u) := diff(r(u), u):$

$$rm(u) = \begin{bmatrix} -\sin(u) \\ \cos(u) \\ 1 \end{bmatrix}$$

$$Jacobi := \sqrt{prik(rm(u), rm(u))} = \sqrt{1 + \sin(u)^2 + \cos(u)^2}$$

$$\int_0^{6\cdot\pi} 1 \cdot Jacobi \, du = 6\sqrt{2}\,\pi$$

Med færdig formel

$$\int_0^{6\cdot\pi} 1 \cdot LinearAlgebra[Norm](diff(r(u), u), 2)\, du = 6\sqrt{2}\,\pi$$

eller

$$\int_0^{6\cdot\pi} 1 \cdot \sqrt{prik(diff(r(u), u), diff(r(u), u))}\, du = 6\sqrt{2}\,\pi$$

Med Integrator8-pakken

$INT := [0, 6\cdot\pi]:$

$kurveIntGo(r, INT, 1) = 6\sqrt{2}\,\pi$

$evalf(\%) = 26.65729763$

Konklusion: kurvelængden er $6\cdot\sqrt{2}\cdot\pi \approx 26.7$

Masse af kurve

Hvis man antager, at kurven har en massetæthed (masse pr. længdeenhed) kaldet f, så kan man beregne kurvens samlede masse.

I dette eksempel afhænger massetætheden alene af z:

$$f(x, y, z) := e^{\frac{z}{20}}:$$

Trinvis

$rm(u) := diff(r(u), u):$

$$rm(u) = \begin{bmatrix} -\sin(u) \\ \cos(u) \\ 1 \end{bmatrix}$$

$$Jacobi := \sqrt{prik(rm(u), rm(u))} = \sqrt{1 + \sin(u)^2 + \cos(u)^2}$$

$$\int_0^{6\cdot\pi} f(vop(r(u))) \cdot Jacobi \, du = -20\sqrt{2} + 20\, e^{\frac{3\pi}{10}}\sqrt{2}$$

Med færdig formel

$$\int_0^{6\cdot\pi} f(vop(r(u))) \cdot LinearAlgebra[Norm](diff(r(u), u), 2)\, du = -20\sqrt{2} + 20\, e^{\frac{3\pi}{10}}\sqrt{2}$$

eller

$$\int_0^{6\cdot\pi} f(vop(r(u))) \cdot \sqrt{prik(diff(r(u), u), diff(r(u), u))}\, du = -20\sqrt{2} + 20\, e^{\frac{3\pi}{10}}\sqrt{2}$$

Med Integrator8-pakken

$INT := [0, 6\cdot\pi]:$

$kurveIntGo(r, INT, f) = 20\sqrt{2}\left(e^{\frac{3\pi}{10}} - 1\right)$

$evalf(\%) = 44.30257034$

Konklusion: kurven masse er ca. 44.3