Fermat-punkt ved sæbeflade

Kilde: "**Agnesis Algebra**" af Aksel Bertelsen, Matematiklærerforeningen, 2012, side 148-151.

http://en.wikipedia.org/wiki/Fermat_point

> *restart*

> $A := [10, 0]; B := [0, 0]; C := [8, 4]; P := [x, y]$

$$A := [10, 0]$$
$$B := [0, 0]$$
$$C := [8, 4]$$
$$P := [x, y] \quad (1)$$

> $AB := \sqrt{(A_1 - B_1)^2 + (A_2 - B_2)^2};$
$BC := \sqrt{(B_1 - C_1)^2 + (B_2 - C_2)^2};$
$AP := \sqrt{(A_1 - P_1)^2 + (A_2 - P_2)^2};$
$BP := \sqrt{(B_1 - P_1)^2 + (B_2 - P_2)^2};$
$CP := \sqrt{(C_1 - P_1)^2 + (C_2 - P_2)^2};$

$$AB := 10$$
$$BC := 4\sqrt{5}$$
$$AP := \sqrt{100 - 20x + x^2 + y^2}$$
$$BP := \sqrt{x^2 + y^2}$$
$$CP := \sqrt{80 - 16x + x^2 - 8y + y^2} \quad (2)$$

2 cosinusrelationer opskrives:

> $Ligning1 := \mathrm{AB}^2 - AP^2 - BP^2 = -2 \cdot \left(-\frac{1}{2}\right) \cdot AP \cdot BP$

$$Ligning1 := 20x - 2x^2 - 2y^2 = \sqrt{100 - 20x + x^2 + y^2}\,\sqrt{x^2 + y^2} \quad (3)$$

> $Ligning2 := BC^2 - BP^2 - CP^2 = -2 \cdot \left(-\frac{1}{2}\right) \cdot BP \cdot CP$

$$Ligning2 := -2x^2 - 2y^2 + 16x + 8y = \sqrt{x^2 + y^2}\,\sqrt{80 - 16x + x^2 - 8y + y^2} \quad (4)$$

Omformes til 2 nye ligninger, så der ikke er kvadratrødder og så højresiden er 0:

> $(lhs(Ligning1))^2 = (rhs(Ligning1))^2;$
$expand(\%);$
$lhs(\%) - rhs(\%) = 0;$
$collect(\%, y);$
$NyLigning1 := \%$

$$(20x - 2x^2 - 2y^2)^2 = (100 - 20x + x^2 + y^2)(x^2 + y^2)$$
$$400x^2 - 80x^3 - 80xy^2 + 4x^4 + 8x^2y^2 + 4y^4 = 100x^2 + 100y^2 - 20x^3 - 20xy^2 + x^4$$

$$+2x^2y^2+y^4$$

$$300x^2-60x^3-60xy^2+3x^4+6x^2y^2+3y^4-100y^2=0$$

$$3y^4+(-60x+6x^2-100)y^2+300x^2-60x^3+3x^4=0$$

$$NyLigning1 := 3y^4+(-60x+6x^2-100)y^2+300x^2-60x^3+3x^4=0 \quad (5)$$

```
> (lhs(Ligning2))^2 = (rhs(Ligning2))^2;
  expand(%);
  lhs(%) - rhs(%) = 0;
  collect(%, y);
  NyLigning2 := %
```

$$(-2x^2-2y^2+16x+8y)^2=(x^2+y^2)(80-16x+x^2-8y+y^2)$$

$$4x^4+8x^2y^2-64x^3-32x^2y+4y^4-64xy^2-32y^3+256x^2+256xy+64y^2=80x^2-16x^3+x^4-8x^2y+2x^2y^2+80y^2-16xy^2-8y^3+y^4$$

$$3x^4+6x^2y^2-48x^3-24x^2y+3y^4-48xy^2-24y^3+176x^2+256xy-16y^2=0$$

$$3y^4-24y^3+(6x^2-48x-16)y^2+(256x-24x^2)y+3x^4-48x^3+176x^2=0$$

$$NyLigning2 := 3y^4-24y^3+(6x^2-48x-16)y^2+(256x-24x^2)y+3x^4-48x^3+176x^2=0 \quad (6)$$

Ved brug af Sylvester matrix:

http://en.wikipedia.org/wiki/Sylvester_matrix

```
> with(LinearAlgebra) :
```

Først struktureres efter y:

```
> Sy := Matrix(8, 8)
```

$$Sy := \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad (7)$$

Koefficienterne struktureret efter y fyldes nu ind i Sylvester matricen:

```
> for j from 1 to 4 do
      for i from 0 to 4 do Sy[j, i + j] := coeff(lhs(NyLigning1), y, 4 - i) end do;
  end do;
  for j from 5 to 8 do
      for i from 0 to 4 do Sy[j, i + j - 4] := coeff(lhs(NyLigning2), y, 4 - i) end do;
  end do;
> Sy
```

$$[[3, 0, -60x+6x^2-100, 0, 300x^2-60x^3+3x^4, 0, 0, 0],$$
$$[0, 3, 0, -60x+6x^2-100, 0, 300x^2-60x^3+3x^4, 0, 0],$$
$$[0, 0, 3, 0, -60x+6x^2-100, 0, 300x^2-60x^3+3x^4, 0],$$
$$[0, 0, 0, 3, 0, -60x+6x^2-100, 0, 300x^2-60x^3+3x^4],$$
$$[3, -24, 6x^2-48x-16, 256x-24x^2, 3x^4-48x^3+176x^2, 0, 0, 0],$$
$$[0, 3, -24, 6x^2-48x-16, 256x-24x^2, 3x^4-48x^3+176x^2, 0, 0],$$
$$[0, 0, 3, -24, 6x^2-48x-16, 256x-24x^2, 3x^4-48x^3+176x^2, 0],$$
$$[0, 0, 0, 3, -24, 6x^2-48x-16, 256x-24x^2, 3x^4-48x^3+176x^2]] \quad \textbf{(8)}$$

> $detSy := Determinant(Sy)$

$$detSy := 196977623040000\,x^4 - 98878095360000\,x^5 + 18203443200000\,x^6 - 1459814400000\,x^7 + 43130880000\,x^8 \quad \textbf{(9)}$$

> $solve(detSy, x); evalf(\%)$

$$0, 0, 0, 0, 8-\frac{4}{3}\sqrt{3}, 8+\frac{4}{3}\sqrt{3}, \frac{116}{13}-\frac{10}{13}\sqrt{3}, \frac{116}{13}+\frac{10}{13}\sqrt{3}$$
$$0., 0., 0., 0., 5.690598923, 10.30940108, 7.590730148, 10.25542370 \quad \textbf{(10)}$$

Den korrekte x-løsning findes her blandt løsningerne, nemlig $x = \frac{116}{13} - \frac{10}{13}\sqrt{3} \approx 7.590730148$

Det er kun $8 - \frac{4}{3}\sqrt{3} = 5.690598923$, som ellers ligger indenfor området mellem de 3 punkter.

> $x1 := \frac{116}{13} - \frac{10}{13}\sqrt{3};$
> $x2 := 8 - \frac{4}{3}\sqrt{3}$

$$x1 := \frac{116}{13} - \frac{10}{13}\sqrt{3}$$
$$x2 := 8 - \frac{4}{3}\sqrt{3} \quad \textbf{(11)}$$

Nu struktureres efter x:

> $(lhs(Ligning1))^2 = (rhs(Ligning1))^2;$
> $expand(\%);$
> $lhs(\%) - rhs(\%) = 0;$
> $collect(\%, x);$
> $NyLigning1 := \%$

$$(20x-2x^2-2y^2)^2 = (100-20x+x^2+y^2)(x^2+y^2)$$
$$400x^2-80x^3-80xy^2+4x^4+8x^2y^2+4y^4 = 100x^2+100y^2-20x^3-20xy^2+x^4+2x^2y^2+y^4$$
$$300x^2-60x^3-60xy^2+3x^4+6x^2y^2+3y^4-100y^2=0$$
$$3x^4-60x^3+(300+6y^2)x^2-60xy^2+3y^4-100y^2=0$$
$$NyLigning1 := 3x^4-60x^3+(300+6y^2)x^2-60xy^2+3y^4-100y^2=0 \quad \textbf{(12)}$$

```
> (lhs(Ligning2))^2 = (rhs(Ligning2))^2;
  expand(%);
  lhs(%) - rhs(%) = 0;
  collect(%, x);
  NyLigning2 := %
```

$$(-2x^2 - 2y^2 + 16x + 8y)^2 = (x^2 + y^2)(80 - 16x + x^2 - 8y + y^2)$$

$$4x^4 + 8x^2y^2 - 64x^3 - 32x^2y + 4y^4 - 64xy^2 - 32y^3 + 256x^2 + 256xy + 64y^2 = 80x^2 - 16x^3 + x^4 - 8x^2y + 2x^2y^2 + 80y^2 - 16xy^2 - 8y^3 + y^4$$

$$3x^4 + 6x^2y^2 - 48x^3 - 24x^2y + 3y^4 - 48xy^2 - 24y^3 + 176x^2 + 256xy - 16y^2 = 0$$

$$3x^4 - 48x^3 + (-24y + 176 + 6y^2)x^2 + (256y - 48y^2)x + 3y^4 - 16y^2 - 24y^3 = 0$$

$$NyLigning2 := 3x^4 - 48x^3 + (-24y + 176 + 6y^2)x^2 + (256y - 48y^2)x + 3y^4 - 16y^2 - 24y^3 = 0 \quad \textbf{(13)}$$

```
> Sx := Matrix(8, 8)
```

$$Sx := \begin{bmatrix} 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0 \end{bmatrix} \quad \textbf{(14)}$$

```
> for j from 1 to 4 do
      for i from 0 to 4 do Sx[j, i+j] := coeff(lhs(NyLigning1), x, 4-i) end do;
  end do;
  for j from 5 to 8 do
      for i from 0 to 4 do Sx[j, i+j-4] := coeff(lhs(NyLigning2), x, 4-i) end do;
  end do;
```

```
> Sx
```

$$[[3, -60, 300 + 6y^2, -60y^2, 3y^4 - 100y^2, 0, 0, 0], \quad \textbf{(15)}$$
$$[0, 3, -60, 300 + 6y^2, -60y^2, 3y^4 - 100y^2, 0, 0],$$
$$[0, 0, 3, -60, 300 + 6y^2, -60y^2, 3y^4 - 100y^2, 0],$$
$$[0, 0, 0, 3, -60, 300 + 6y^2, -60y^2, 3y^4 - 100y^2],$$
$$[3, -48, -24y + 176 + 6y^2, 256y - 48y^2, 3y^4 - 16y^2 - 24y^3, 0, 0, 0],$$
$$[0, 3, -48, -24y + 176 + 6y^2, 256y - 48y^2, 3y^4 - 16y^2 - 24y^3, 0, 0],$$
$$[0, 0, 3, -48, -24y + 176 + 6y^2, 256y - 48y^2, 3y^4 - 16y^2 - 24y^3, 0],$$
$$[0, 0, 0, 3, -48, -24y + 176 + 6y^2, 256y - 48y^2, 3y^4 - 16y^2 - 24y^3]]$$

```
> detSx := Determinant(Sx)
```

$$detSx := -259522560000\,y^4 + 247726080000\,y^5 + 774144000000\,y^6 - 464486400000\,y^7 \quad \textbf{(16)}$$

$+ 43130880000\,y^8$

> *solve*(*detSx*, *y*); *evalf*(%)

$$0, 0, 0, 0, 4 - \frac{8}{3}\sqrt{3}, 4 + \frac{8}{3}\sqrt{3}, \frac{18}{13} - \frac{20}{39}\sqrt{3}, \frac{18}{13} + \frac{20}{39}\sqrt{3}$$

$$0., 0., 0., 0., -0.618802155, 8.618802155, 0.4963842014, 2.272846569 \quad \textbf{(17)}$$

Den korrekte y-løsning findes her blandt løsningerne, nemlig $y = \frac{18}{13} + \frac{20}{39}\sqrt{3} \approx 2.272846569$

Det er kun $\frac{18}{13} - \frac{20}{39}\sqrt{3} = 0.4963842014$, som ellers ligger indenfor området mellem de 3 punkter.

> $y1 := \frac{18}{13} + \frac{20}{39}\sqrt{3}$;
$y2 := \frac{18}{13} - \frac{20}{39}\sqrt{3}$

$$y1 := \frac{18}{13} + \frac{20}{39}\sqrt{3}$$

$$y2 := \frac{18}{13} - \frac{20}{39}\sqrt{3} \quad \textbf{(18)}$$

Check af mulighederne. Gøre prøven ved at indsætte de mulige kombinationer af x og y :

> *subs*($x = x1$, $y = y1$, *Ligning1*) : *simplify*(%); *evalf*(%);
subs($x = x1$, $y = y1$, *Ligning2*) : *simplify*(%); *evalf*(%)

$$\frac{400}{39} + \frac{120}{13}\sqrt{3} = \frac{400}{39} + \frac{120}{13}\sqrt{3}$$

$$26.24457157 = 26.24457157$$

$$-\frac{560}{39} + \frac{640}{39}\sqrt{3} = -\frac{560}{39} + \frac{640}{39}\sqrt{3}$$

$$14.06442351 = 14.06442351 \quad \textbf{(19)}$$

$(x1, y1)$ virker!

> *subs*($x = x2$, $y = y1$, *Ligning1*) : *simplify*(%); *evalf*(%);
subs($x = x2$, $y = y1$, *Ligning2*) : *simplify*(%); *evalf*(%)

$$\frac{8072}{507} + \frac{2224}{169}\sqrt{3} = \frac{4}{1521}\sqrt{4578 + 2568\sqrt{3}}\,\sqrt{27393 - 7572\sqrt{3}}$$

$$38.71448321 = 29.85441265$$

$$-\frac{2536}{507} + \frac{11456}{507}\sqrt{3} = \frac{4}{1521}\sqrt{27393 - 7572\sqrt{3}}\,\sqrt{4929 - 1020\sqrt{3}}$$

$$34.13486007 = 17.67116624 \quad \textbf{(20)}$$

$(x2, y1)$ dur ikke!

> *subs*($x = x1$, $y = y2$, *Ligning1*) : *simplify*(%); *evalf*(%);
subs($x = x1$, $y = y2$, *Ligning2*) : *simplify*(%); *evalf*(%)

$$\frac{400}{39} + \frac{2520}{169}\sqrt{3} = \frac{8}{1521}\sqrt{2145 + 90\sqrt{3}}\,\sqrt{7995 - 1440\sqrt{3}}$$

$$36.08344005 = 18.71211877$$

$$-\frac{560}{39}+\frac{7040}{507}\sqrt{3}=\frac{16}{1521}\sqrt{7995-1440\sqrt{3}}\sqrt{975+120\sqrt{3}}$$

$$9.69159308=26.83302686 \quad \textbf{(21)}$$

$(x1, y2)$ dur ikke!

> *subs*$(x=x2, y=y2, Ligning1)$: *simplify*(%); *evalf*(%);
> *subs*$(x=x2, y=y2, Ligning2)$: *simplify*(%); *evalf*(%)

$$\frac{8072}{507}+\frac{3184}{169}\sqrt{3}=\frac{4}{1521}\sqrt{4578+1488\sqrt{3}}\sqrt{27393-8652\sqrt{3}}$$

$$48.55335172=24.77895717$$

$$-\frac{2536}{507}+\frac{3392}{169}\sqrt{3}=\frac{4}{1521}\sqrt{27393-8652\sqrt{3}}\sqrt{4929+1020\sqrt{3}}$$

$$29.76202963=23.96994759 \quad \textbf{(22)}$$

$(x2, y2)$ dur ikke!

Konklusion: Det er hermed vist, at Fermat-punktet ligger i

$$P=(x1, y1)=\left(\frac{116}{13}-\frac{10}{13}\sqrt{3}, \frac{18}{13}+\frac{20}{39}\sqrt{3}\right)\approx(7.590730148, 2.272846569)$$