Sylvester matrix

Kilde: "**Agnesis Algebra**" af Aksel Bertelsen, Matematiklærerforeningen, 2012, side 146-148.

http://en.wikipedia.org/wiki/Sylvester_matrix

```
> restart
```

```
> S := Matrix([[1, a, b, 0], [0, 1, a, b], [1, c, d, 0], [0, 1, c, d]])
```

$$S := \begin{bmatrix} 1 & a & b & 0 \\ 0 & 1 & a & b \\ 1 & c & d & 0 \\ 0 & 1 & c & d \end{bmatrix} \tag{1}$$

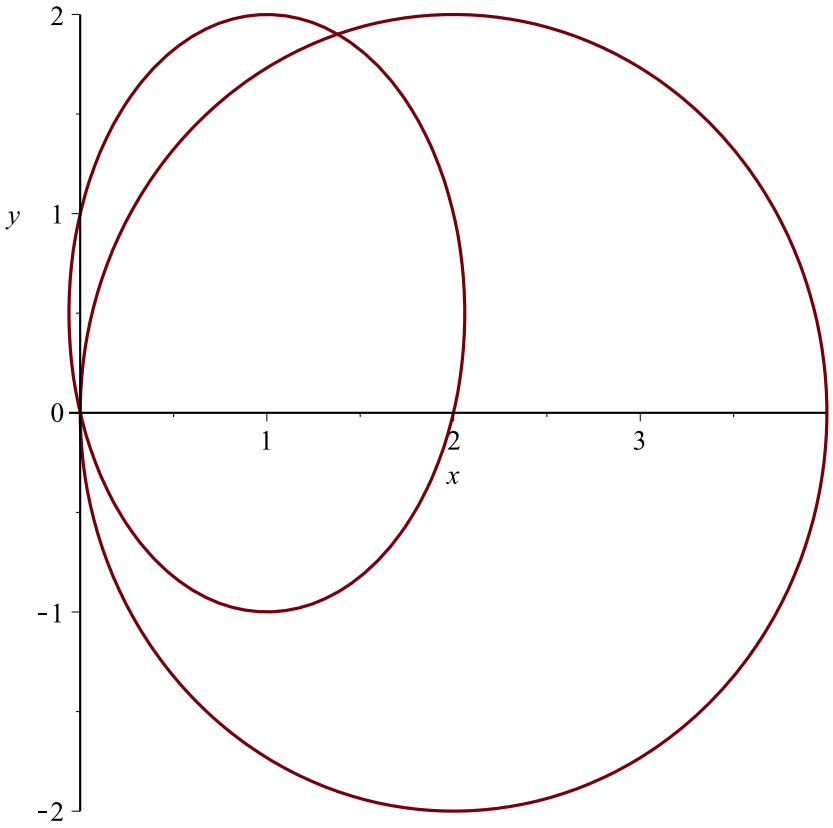
```
> with(LinearAlgebra) :
> detS := Determinant(S)
```

$$detS := d^2 + c^2 b - c a d - 2 d b + a^2 d - a c b + b^2 \tag{2}$$

```
> Ligning1 := y^2 + x^2 - 4*x = 0;
  Ligning2 := y^2 - y + 2*x^2 - 4*x = 0
```

$$Ligning1 := y^2 + x^2 - 4 x = 0$$
$$Ligning2 := y^2 - y + 2 x^2 - 4 x = 0 \tag{3}$$

```
> with(plots) :
  implicitplot({Ligning1, Ligning2}, x = -5..5, y = -5..5, numpoints = 20000)
```



Der ser ud til at være 2 skæringspunkter!

Direkte løsning:

> $solve(\{Ligning1, Ligning2\}); evalf(\%)$

$$\{x=0, y=0\}, \left\{x=RootOf\left(\_Z-4+\_Z^3\right), y=RootOf\left(\_Z-4+\_Z^3\right)^2\right\}$$
$$\{x=0., y=0.\}, \{x=1.37879670012955, y=1.90108034028814\} \quad \textbf{(4)}$$

> $solve(\{z-4+z^3=0, z>0\}); evalf(\%)$

$$\left\{z=\frac{1}{3}\left(54+3\sqrt{327}\right)^{1/3}-\frac{1}{\left(54+3\sqrt{327}\right)^{1/3}}\right\}$$
$$\{z=1.378796700\} \quad \textbf{(5)}$$

Omskrivning af de 2 ligninger til Sylvester matrix:

> $a := 0; b := x^2-4\cdot x; c := -1; d := 2\cdot x^2-4\cdot x$

$$a := 0$$
$$b := x^2-4x$$
$$c := -1$$
$$d := 2x^2-4x \quad \textbf{(6)}$$

> S

$$\begin{bmatrix} 1 & 0 & x^2-4x & 0 \\ 0 & 1 & 0 & x^2-4x \\ 1 & -1 & 2x^2-4x & 0 \\ 0 & 1 & -1 & 2x^2-4x \end{bmatrix} \quad (7)$$

> $detS$

$$(2x^2-4x)^2+x^2-4x-2(2x^2-4x)(x^2-4x)+(x^2-4x)^2 \quad (8)$$

> $simplify(detS)$

$$x^2-4x+x^4 \quad (9)$$

Alle (komplekse) rødder i polynomiet findes:

> $solve(\text{detS}); evalf(\%)$

$$0, \frac{1}{3}(54+3\sqrt{327})^{1/3}-\frac{1}{(54+3\sqrt{327})^{1/3}}, -\frac{1}{6}(54+3\sqrt{327})^{1/3}+\frac{1}{2(54+3\sqrt{327})^{1/3}}+\frac{1}{2}I\sqrt{3}\left(\frac{1}{3}(54+3\sqrt{327})^{1/3}+\frac{1}{(54+3\sqrt{327})^{1/3}}\right), -\frac{1}{6}(54+3\sqrt{327})^{1/3}+\frac{1}{2(54+3\sqrt{327})^{1/3}}-\frac{1}{2}I\sqrt{3}\left(\frac{1}{3}(54+3\sqrt{327})^{1/3}+\frac{1}{(54+3\sqrt{327})^{1/3}}\right)$$

$$0., 1.378796700, -0.6893983505+1.557501287\,I, -0.6893983505-1.557501287\,I \quad (10)$$

Alle **reelle** rødder i polynomiet findes:

> $with(RealDomain):$
> $solve(detS);$
> $evalf(\%);$
> $unwith(RealDomain)$

$$\frac{1}{3}(54+3\sqrt{327})^{1/3}-\frac{1}{(54+3\sqrt{327})^{1/3}}, 0$$

$$1.378796700, 0. \quad (11)$$

> $x1 := (11)[1];$
> $x2 := (11)[2]$

$$x1 := 1.378796700$$
$$x2 := 0. \quad (12)$$

> $solve(subs(x=x1, Ligning1), y);$
> $solve(subs(x=x1, Ligning2), y)$

$$1.901080340, -1.901080340$$
$$1.901080340, -0.9010803403 \quad (13)$$

> $y1 := \%[1]$

$$y1 := 1.901080340 \quad (14)$$

Dvs. $(x, y) = (1.378796700, 1.901080340)$ er et skæringspunkt.

```
> solve(subs(x = x2, Ligning1), y);
  solve(subs(x = x2, Ligning2), y)
```

$$0, 0$$
$$0, 1 \qquad \textbf{(15)}$$

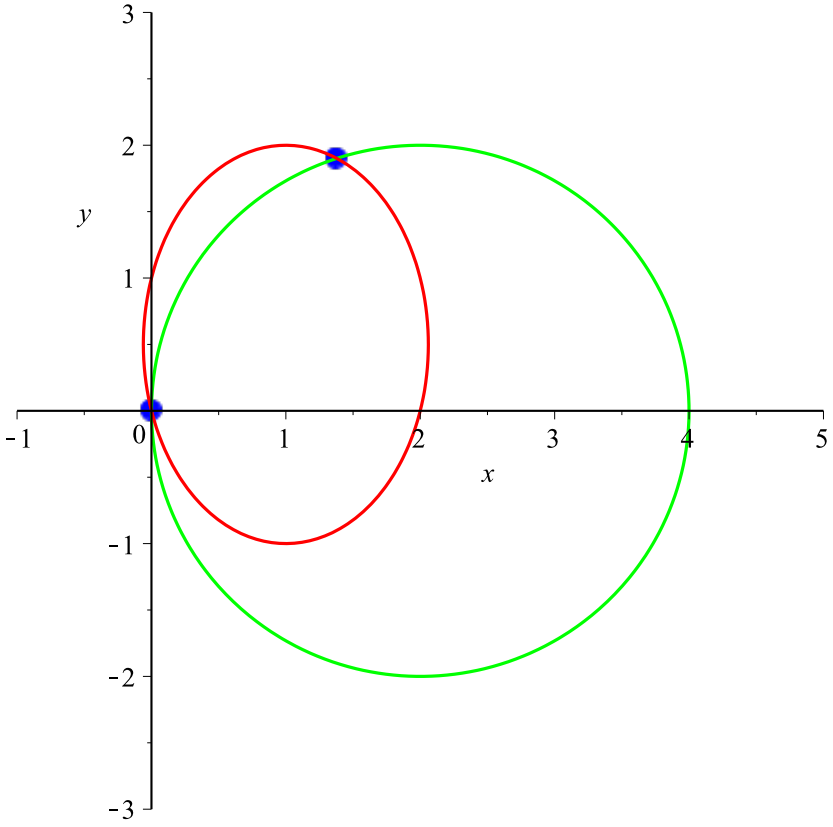
Dvs. $(x, y) = (0, 0)$ er et skæringspunkt.

```
> y2 := %[1]
```

$$y2 := 0 \qquad \textbf{(16)}$$

Graf med skæringspunkter indtegnet, og lige store enheder på akserne:

```
> Punkter := pointplot({[x1, y1], [x2, y2]}, color = blue, symbolsize = 20, symbol = solidcircle, ) :
  Kurve1 := implicitplot(Ligning1, x = -2 ..6, y = -2 ..6, numpoints = 20000, color = green) :
  Kurve2 := implicitplot(Ligning2, x = -2 ..6, y = -2 ..6, numpoints = 20000, color = red) :
  display(Punkter, Kurve1, Kurve2, view = [-1 ..5, -3 ..3])
```



Konklusion: Ellipsen og cirklen har 2 skæringspunkter, nemlig $(1.378796700, 1.901080340)$ og $(0, 0)$

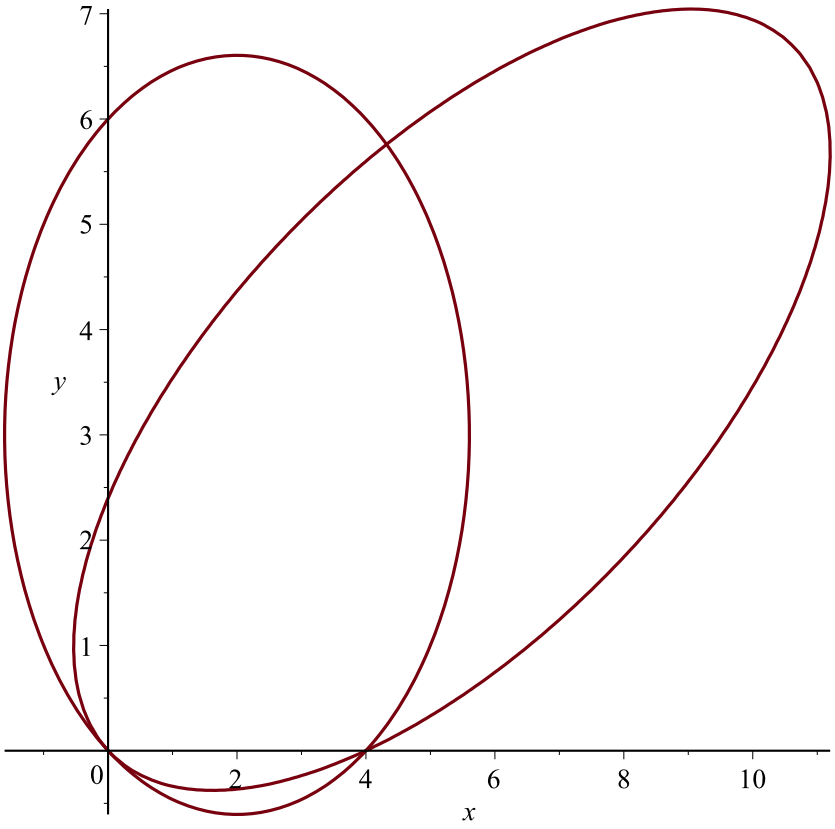
Skæring mellem cirkel og ellipse (eksempel)

```
> restart
> with(LinearAlgebra) :
> Ligning1 := 2·x^2 − 4·x·y + 5·y^2 − 8·x − 12·y = 0;
  Ligning2 := x^2 − 4·x + y^2 − 6·y = 0
```

$$Ligning1 := 2x^2 - 4xy + 5y^2 - 8x - 12y = 0$$

$$Ligning2 := x^2 - 4x + y^2 - 6y = 0 \qquad \textbf{(1.1)}$$

```
> with(plots) :
  implicitplot({Ligning1, Ligning2}, x = -5..15, y = -5..15, numpoints = 20000)
```



Der ser ud til at være 3 skæringspunkter!

Ved brug af Sylvester matrix:

```
> collect(Ligning1, y);
  collect(Ligning2, y)
```

$$5y^2 + (-4x - 12)y + 2x^2 - 8x = 0$$

$$x^2 - 4x + y^2 - 6y = 0 \qquad \textbf{(1.2)}$$

> $S := \begin{bmatrix} 5 & -4\cdot x-12 & 2\cdot x^2-8\cdot x & 0 \\ 0 & 5 & -4\cdot x-12 & 2\cdot x^2-8\cdot x \\ 1 & -6 & x^2-4\cdot x & 0 \\ 0 & 1 & -6 & x^2-4\cdot x \end{bmatrix}$

$$S := \begin{bmatrix} 5 & -4x-12 & 2x^2-8x & 0 \\ 0 & 5 & -4x-12 & 2x^2-8x \\ 1 & -6 & x^2-4x & 0 \\ 0 & 1 & -6 & x^2-4x \end{bmatrix} \quad \textbf{(1.3)}$$

> $detS := Determinant(S)$

$$detS := 25x^4-208x^3+432x^2 \quad \textbf{(1.4)}$$

> $solve(detS=0, x); evalf(\%)$

$$0, 0, \frac{108}{25}, 4$$

$$0., 0., 4.320000000, 4. \quad \textbf{(1.5)}$$

> $solve(subs(x=0, Ligning1), y);$
> $solve(subs(x=0, Ligning2), y)$

$$0, \frac{12}{5}$$

$$0, 6 \quad \textbf{(1.6)}$$

Dvs. løsning bliver $\underline{(x, y) = (0, 0)}$

> $solve(subs(x=4, Ligning1), y);$
> $solve(subs(x=4, Ligning2), y)$

$$0, \frac{28}{5}$$

$$0, 6 \quad \textbf{(1.7)}$$

Dvs. løsning bliver $\underline{(x, y) = (4, 0)}$

> $solve(subs(x=4.32, Ligning1), y);$
> $solve(subs(x=4.32, Ligning2), y)$

$$5.760000000, 0.09600000000$$

$$5.760000000, 0.2400000000 \quad \textbf{(1.8)}$$

Dvs. løsning bliver $\underline{(x, y) = (4.32, 5.76)}$

Direkte løsning af 2 ligninger med 2 ubekendte i Maple:

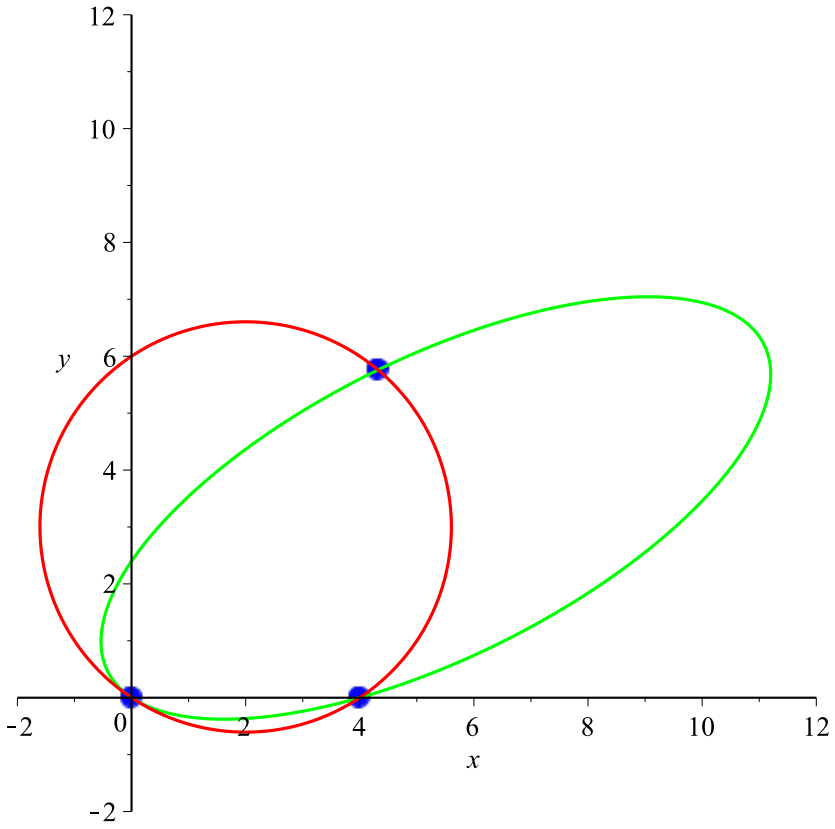
> $solve(\{Ligning1, Ligning2\}); evalf(\%)$

$$\left\{x=\frac{108}{25}, y=\frac{144}{25}\right\}, \{x=0, y=0\}, \{x=4, y=0\}$$

$$\{x=4.320000000, y=5.760000000\}, \{x=0., y=0.\}, \{x=4., y=0.\} \quad \textbf{(1.9)}$$

Graf med skæringspunkter indtegnet, og lige store enheder på akserne:

```
> with(plots) :
  Punkter := pointplot( {[0, 0], [4, 0], [4.32, 5.76]}, color = blue, symbolsize = 20, symbol
       = solidcircle, ) :
  Kurve1 := implicitplot(Ligning1, x = -5 ..12, y = -2 ..12, numpoints = 20000, color = green) :
  Kurve2 := implicitplot(Ligning2, x = -5 ..12, y = -2 ..12, numpoints = 20000, color = red) :
  display(Punkter, Kurve1, Kurve2, view = [ -2 ..12, -2 ..12 ])
```



Konklusion: Ellipsen og cirklen har 3 skæringspunkter, nemlig (0, 0) , (4, 0) og (4.32, 5.76)