Arealer som summer

Numerisk integration

http://www.zweigmedia2.com/RealWorld/integral/numint.html

Her kan ses formlerne, som er implementeret nedenfor med en effektiv kode.

Antag, at funktionen f er **positiv**, og man ønsker arealet under grafen fra a til b .

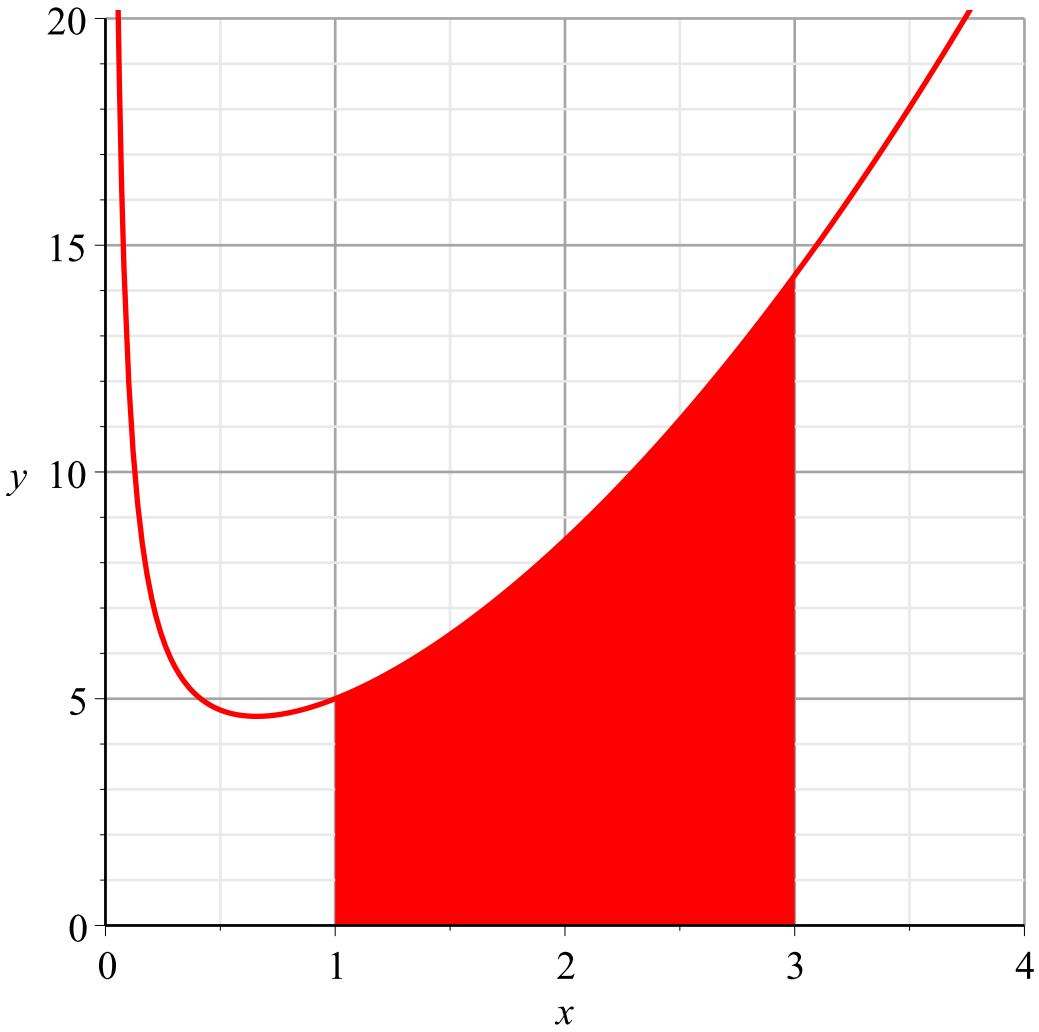
```
> restart
> with(plots) : with(Student[Calculus1]) :
> a := 1; b := 3; f := x→x^2 + x + 2 + 1/x
```

$$a := 1$$

$$b := 3$$

$$f := x \to x^2 + x + 2 + \frac{1}{x} \quad \textbf{(1)}$$

```
> graf := plot(f(x), x = 0..4, y = 0..20, gridlines, color = red, thickness = 2) :
  område := plot(f(x), x = 1..3, y = 0..20, filled, color = red) :
  display(område, graf)
```



Man kan bevise, at fejlen i de forskellige metoder er:

Højresum og venstresum: fejlen går som n^{-1} (dvs. hvis $n=100$, så er fejlen af størrelsesorden $100^{-1}=10^{-2}=0.01$)
Midtsum og trapezsum: fejlen går som n^{-2} (dvs. hvis $n=100$, så er fejlen af størrelsesorden $100^{-2}=10^{-4}=0.0001$)
Simpsonsum: fejlen går som n^{-4} (dvs. hvis $n=100$, så er fejlen af størrelsesorden $100^{-4}=10^{-8}=0.00000001$)

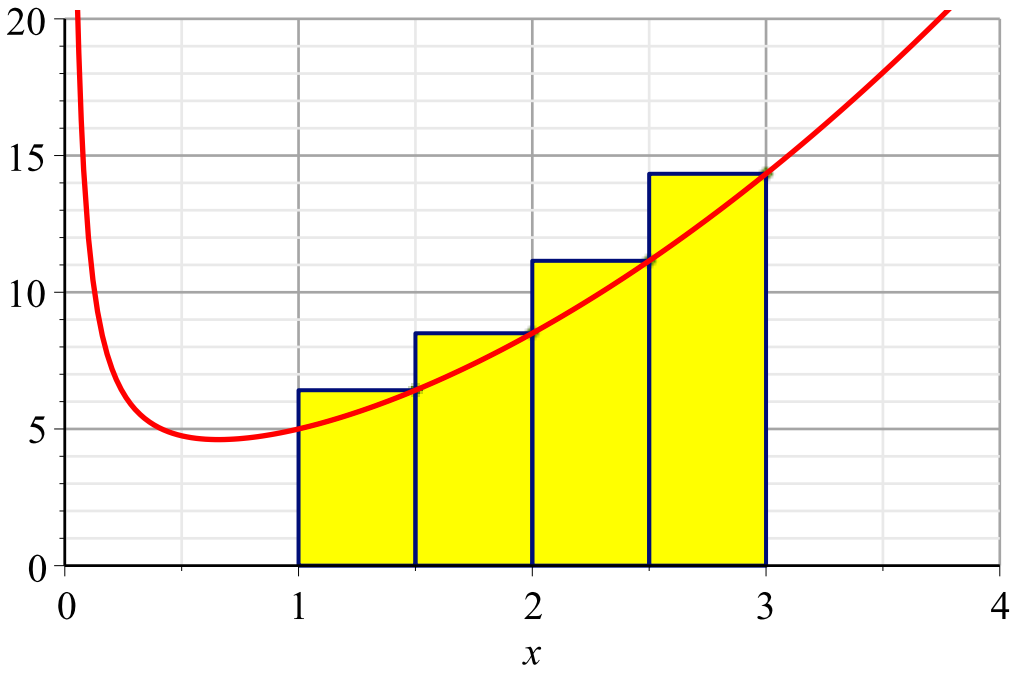
Det betyder, at man i praksis anvender **Simpsonsum**, når et bestemt integral skal beregnes tilnærmet (numerisk).

Højresum (højre endepunkt i hvert interval)

Formel med n inddelinger:

$$\textbf{Højre}=\left(\sum_{i=1}^{n} f\left(a+i\cdot\frac{b-a}{n}\right)\right)\cdot\left(\frac{b-a}{n}\right)$$

```
> opdeling := ApproximateInt(f(x), 1..3, method=right, output=plot, partition=4,
      boxoptions=[filled=[color=yellow, transparency=0]]) :
  display(opdeling, graf)
```



A right Riemann sum approximation of $\int_1^3 f(x)\,dx$, where $f(x) = x^2 + x + 2 + \frac{1}{x}$ and the partition is uniform. The approximate value of the integral is 20.20000000. Number of subintervals used: 4.

> $Areal := \int_1^3 f(x)\,dx;\ evalf(\%)$

$$Areal := \frac{50}{3} + \ln(3)$$

$$17.76527896 \qquad \textbf{(1.1)}$$

> $H := n \to \left(\sum_{i=1}^{n} f\left(a + i \cdot \frac{b-a}{n} \right) \right) \cdot \left(\frac{b-a}{n} \right)$:

> $[n = 4, \text{'Højresum'} = evalf(H(4))]$;
$[n = 40, \text{'Højresum'} = evalf(H(40))]$;
$[n = 400, \text{'Højresum'} = evalf(H(400))]$;
$[n = 4000, \text{'Højresum'} = evalf(H(4000))]$;
$[n = 40000, \text{'Højresum'} = evalf(H(40000))]$;
$[n = 400000, \text{'Højresum'} = evalf(H(400000))]$;
$[n = 4000000, \text{'Højresum'} = evalf(H(4000000))]$;
$[n = 40000000, \text{'Højresum'} = evalf(H(40000000))]$;
$[n = 400000000, \text{'Højresum'} = evalf(H(400000000))]$;
$[n = 4000000000, \text{'Højresum'} = evalf(H(4000000000))]$

$$[n = 4, Højresum = 20.20000000]$$
$$[n = 40, Højresum = 17.99963076]$$
$$[n = 400, Højresum = 17.78862247]$$

[$n = 4000$, *Højresum* = 17.76761239]
[$n = 40000$, *Højresum* = 17.76551229]
[$n = 400000$, *Højresum* = 17.76530229]
[$n = 4000000$, *Højresum* = 17.76528129]
[$n = 40000000$, *Højresum* = 17.76527919]
[$n = 400000000$, *Højresum* = 17.76527897]
[$n = 4000000000$, *Højresum* = 17.76527896] **(1.2)**

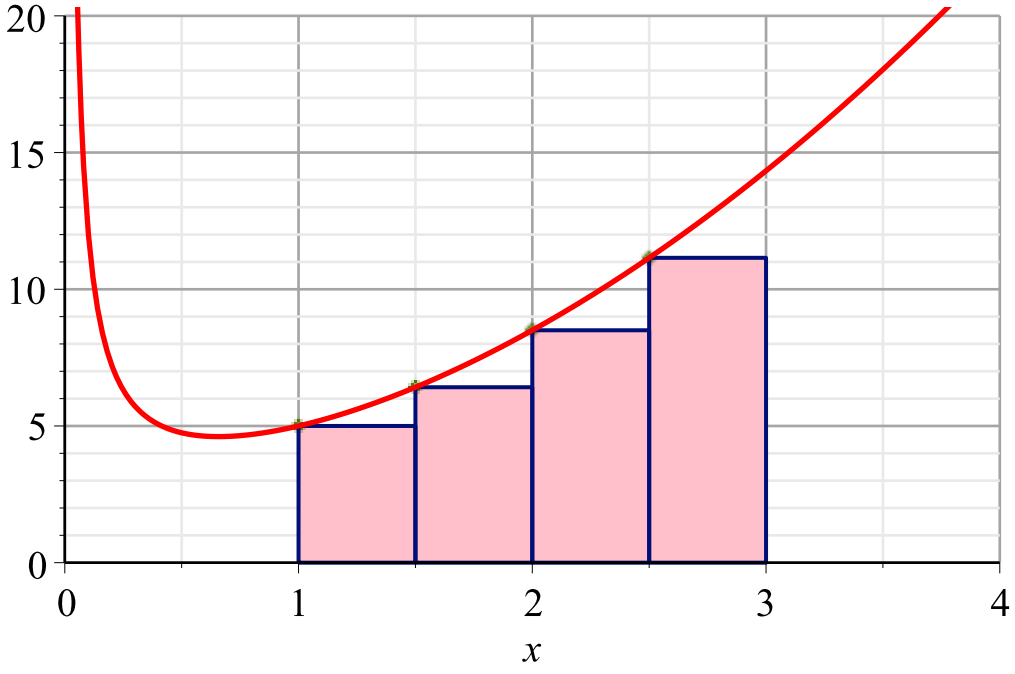
Højresummerne giver et for stort resultat, fordi funktionen er voksende.
For et meget stort antal inddelinger vil højresummen blive = arealet.

Venstresum (venstre endepunkt i hvert interval)

Formel med n inddelinger:

$$Venstre = \left(\sum_{i=0}^{n-1} f\left(a + i \cdot \frac{b-a}{n} \right) \right) \cdot \left(\frac{b-a}{n} \right)$$

```
> opdeling := ApproximateInt(f(x), 1..3, method=left, output=plot, partition=4,
      boxoptions=[filled=[color=pink, transparency=0]]):
  display(opdeling, graf)
```



A left Riemann sum approximation of $\int_1^3 f(x)\,dx$, where $f(x) = x^2 + x + 2 + \frac{1}{x}$ and the partition is uniform. The approximate value of the integral is 15.53333333. Number of subintervals used: 4.

> $Areal := \int_1^3 f(x)\,dx;\ evalf(\%)$

$$Areal := \frac{50}{3} + \ln(3)$$

$$17.76527896 \qquad \textbf{(2.1)}$$

> $V := n \to \left(\sum_{i=0}^{n-1} f\left(a + i \cdot \frac{b-a}{n} \right) \right) \cdot \left(\frac{b-a}{n} \right) :$

> $[n = 4, 'Venstresum' = evalf(V(4))];$
> $[n = 40, 'Venstresum' = evalf(V(40))];$
> $[n = 400, 'Venstresum' = evalf(V(400))];$
> $[n = 4000, 'Venstresum' = evalf(V(4000))];$
> $[n = 40000, 'Venstresum' = evalf(V(40000))];$
> $[n = 400000, 'Venstresum' = evalf(V(400000))];$
> $[n = 4000000, 'Venstresum' = evalf(V(4000000))];$
> $[n = 40000000, 'Venstresum' = evalf(V(40000000))];$
> $[n = 400000000, 'Venstresum' = evalf(V(400000000))];$
> $[n = 4000000000, 'Venstresum' = evalf(V(4000000000))];$
> $[n = 40000000000, 'Venstresum' = evalf(V(40000000000))]$

$$[n = 4, Venstresum = 15.53333333]$$
$$[n = 40, Venstresum = 17.53296409]$$
$$[n = 400, Venstresum = 17.74195581]$$
$$[n = 4000, Venstresum = 17.76294572]$$
$$[n = 40000, Venstresum = 17.76504563]$$
$$[n = 400000, Venstresum = 17.76525562]$$
$$[n = 4000000, Venstresum = 17.76527662]$$
$$[n = 40000000, Venstresum = 17.76527872]$$
$$[n = 400000000, Venstresum = 17.76527893]$$
$$[n = 4000000000, Venstresum = 17.76527895]$$
$$[n = 40000000000, Venstresum = 17.76527896] \qquad \textbf{(2.2)}$$

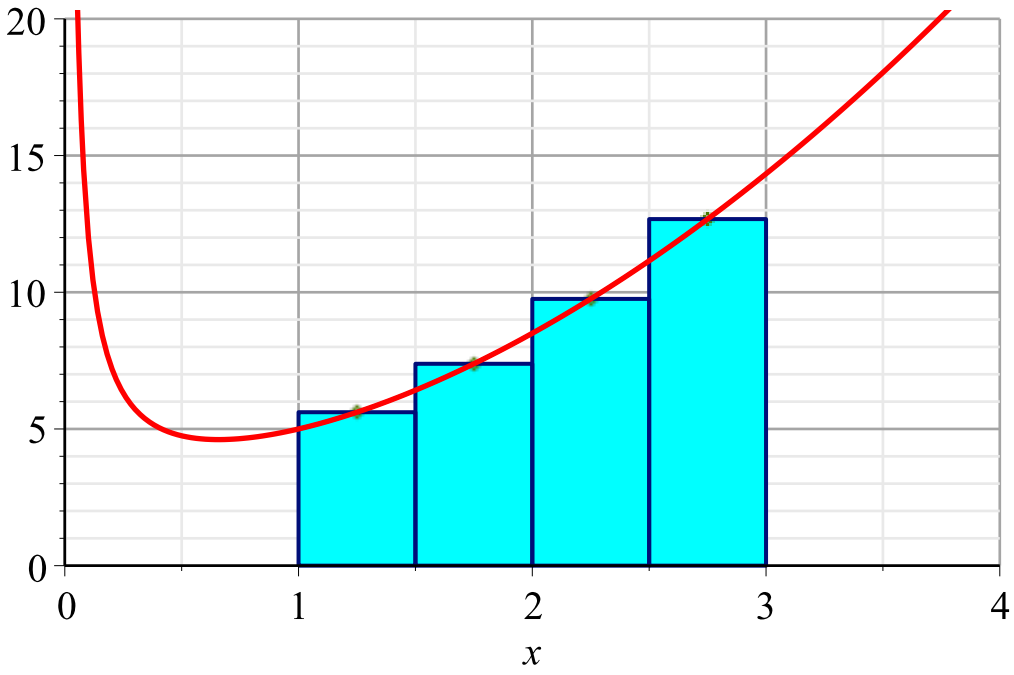
Venstresummerne giver et for lille resultat, fordi funktionen er voksende.
For et meget stort antal inddelinger vil venstresummen blive = arealet.

Midtsum (midtpunkt i hvert interval)

Formel med n inddelinger:

$$\textbf{Midt} = \left(\sum_{i=0}^{n-1} f\left(a + \left(i + \frac{1}{2} \right) \cdot \frac{b-a}{n} \right) \right) \cdot \left(\frac{b-a}{n} \right)$$

> $opdeling := ApproximateInt(f(x), 1..3, \textbf{method = midpoint}, output = plot, partition = 4,$
> $boxoptions = [filled = [\textbf{color = cyan}, transparency = 0]]) :$
> $display(opdeling, graf)$



A midpoint Riemann sum approximation of $\int_1^3 f(x)\,dx$, where $f(x) = x^2 + x + 2 + \frac{1}{x}$ and the partition is uniform. The approximate value of the integral is 17.71475469. Number of subintervals used: 4.

> $Areal := \int_1^3 f(x)\,dx;\ evalf(\%)$

$$Areal := \frac{50}{3} + \ln(3)$$

$$17.76527896 \qquad \textbf{(3.1)}$$

> $M := n \to \left(\sum_{i=0}^{n-1} f\left(a + \left(i + \frac{1}{2} \right) \cdot \frac{b-a}{n} \right) \right) \cdot \left(\frac{b-a}{n} \right) :$

> [n = 4, 'Midtsum' = evalf(M(4))];
[n = 40, 'Midtsum' = evalf(M(40))];
[n = 400, 'Midtsum' = evalf(M(400))];
[n = 4000, 'Midtsum' = evalf(M(4000))];
[n = 40000, 'Midtsum' = evalf(M(40000))]

$$[n = 4,\ Midtsum = 17.71475469]$$
$$[n = 40,\ Midtsum = 17.76476974]$$
$$[n = 400,\ Midtsum = 17.76527386]$$
$$[n = 4000,\ Midtsum = 17.76527890]$$
$$[n = 40000,\ Midtsum = 17.76527896] \qquad \textbf{(3.2)}$$

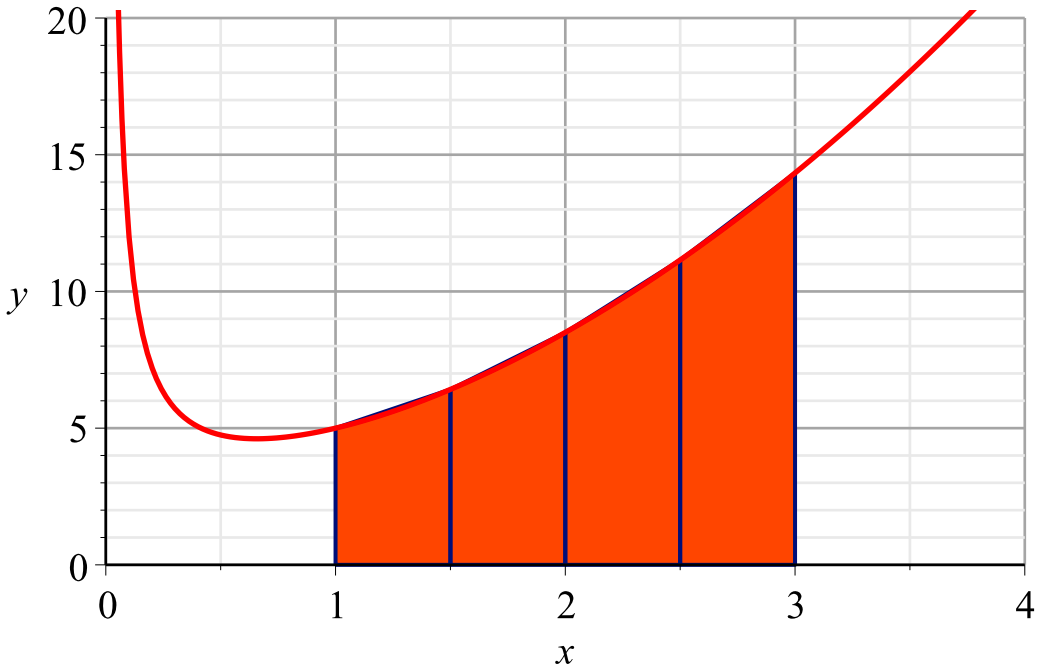
Midtsum giver et resultat, som er tættere på arealet.
Man behøver ikke så mange inddelinger som ved venstre- eller højresum.

Trapezsum (ret linje gennem 2 nabopunkter)

Formel med n inddelinger:

$$Trapez = \left(\frac{f(a)}{2} + \sum_{i=1}^{n-1} f\left(a + i \cdot \frac{b-a}{n}\right) + \frac{f(b)}{2} \right) \cdot \left(\frac{b-a}{n} \right)$$

```
> opdeling := ApproximateInt(f(x), 1..3, method=trapezoid, output=plot, partition=4,
      boxoptions=[filled=[color="OrangeRed",transparency=0]]) :
  display(opdeling, graf)
```



An approximation of $\int_1^3 f(x)\,dx$ using trapezoid rule, where $f(x) = x^2 + x + 2 + \frac{1}{x}$ and the partition is uniform. The approximate value of the integral is 17.86666667. Number of subintervals used: 4.

```
> Areal := int(f(x), x=1..3); evalf(%)
```

$$Areal := \frac{50}{3} + \ln(3)$$

$$17.76527896 \qquad \textbf{(4.1)}$$

```
> T := n -> ( (f(a)/2 + sum(f(a + i*(b-a)/n), i=1..n-1) + f(b)/2) * ((b-a)/n) ) :
```

```
> [n=4, 'Trapezsum'=evalf(T(4))];
  [n=40, 'Trapezsum'=evalf(T(40))];
  [n=400, 'Trapezsum'=evalf(T(400))];
```

$[n = 4000, \text{'Trapezsum'} = evalf(T(4000))];$
$[n = 40000, \text{'Trapezsum'} = evalf(T(40000))]$

$$[n = 4,\ Trapezsum = 17.86666667]$$
$$[n = 40,\ Trapezsum = 17.76629742]$$
$$[n = 400,\ Trapezsum = 17.76528914]$$
$$[n = 4000,\ Trapezsum = 17.76527905]$$
$$[n = 40000,\ Trapezsum = 17.76527896] \quad \textbf{(4.2)}$$

Trapezsum giver et resultat, som er tættere på arealet.
Man behøver ikke så mange inddelinger som ved venstre- eller højresum.

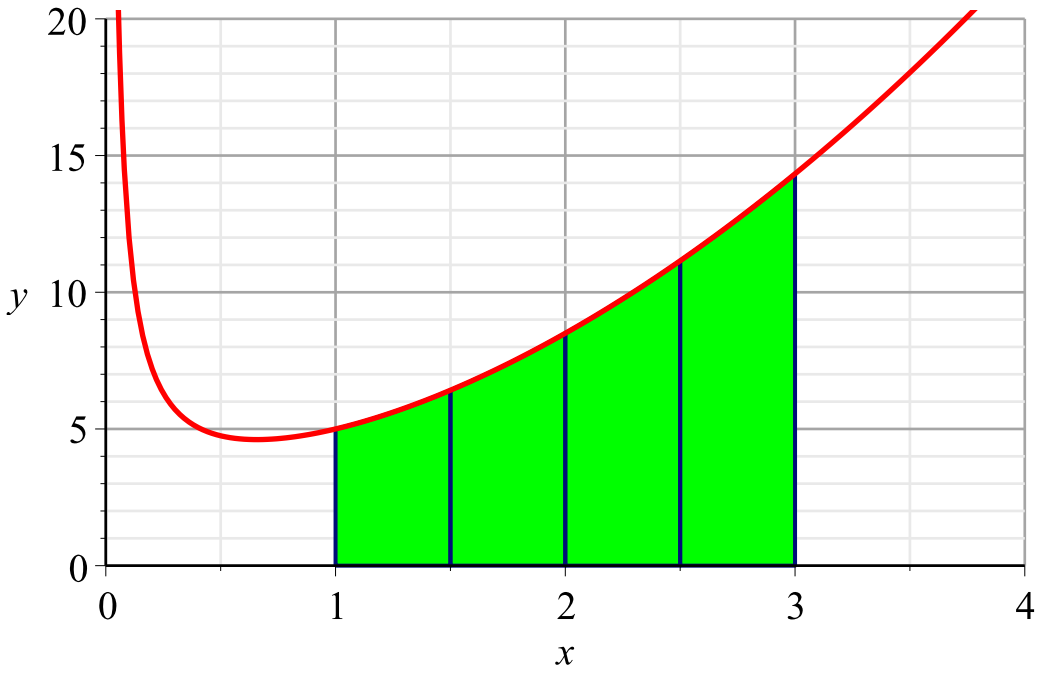
Simpsonsum (parabler gennem 3 nabopunkter)

Antag, at *n* er et LIGE tal.

Formel med *n* inddelinger:

$$Simpson = \frac{1}{3} \cdot \left(f(a) + 4 \cdot \left(\sum_{i=1}^{\frac{n}{2}} f\left(a + (2 \cdot i - 1) \cdot \frac{b-a}{n} \right) \right) + 2 \cdot \left(\sum_{i=1}^{\frac{n}{2}-1} f\left(a + 2 \cdot i \cdot \frac{b-a}{n} \right) \right) + f(b) \right) \cdot \left(\frac{b-a}{n} \right)$$

```
> opdeling := ApproximateInt(f(x), 1..3, method=simpson, output=plot, partition=4,
      boxoptions=[filled=[color=green, transparency=0]]):
  display(opdeling, graf)
```



An approximation of $\int_1^3 f(x)\,dx$ using Simpson's rule, where $f(x) = x^2 + x + 2 + \frac{1}{x}$ and the partition is uniform. The approximate value of the integral is 17.76539202. Number of subintervals used: 4.

> $Areal := \int_1^3 f(x)\,dx;\ evalf(\%)$

$$Areal := \frac{50}{3} + \ln(3)$$

$$17.76527896 \qquad \textbf{(5.1)}$$

> $S := n \to \frac{1}{3} \cdot \left(f(a) + 4 \cdot \left(\sum_{i=1}^{\frac{n}{2}} f\left(a + (2 \cdot i - 1) \cdot \frac{b-a}{n} \right) \right) + 2 \cdot \left(\sum_{i=1}^{\frac{n}{2}-1} f\left(a + 2 \cdot i \cdot \frac{b-a}{n} \right) \right) + f(b) \right) \cdot \left(\frac{b-a}{n} \right)$:

> $[n = 4, \text{'Simpsonsum'} = evalf(S(4))];$
> $[n = 40, \text{'Simpsonsum'} = evalf(S(40))];$
> $[n = 400, \text{'Simpsonsum'} = evalf(S(400))]$

$$[n = 4,\ Simpsonsum = 17.76666667]$$
$$[n = 40,\ Simpsonsum = 17.76527916]$$
$$[n = 400,\ Simpsonsum = 17.76527896] \qquad \textbf{(5.2)}$$

Simpsonsum giver et resultat, som er meget tæt på arealet.
Man behøver ikke så mange inddelinger som ved midt- eller trapezsum.

NB: Simpsonsum giver eksakt værdi på et 2. grads polynomium!

Formel for Simpsonsum udtrykt ved midtsum og trapezsum
Test med givet $f(x)$

Tak til Lars-Lund-Hansen, som er kommet med dette indspark.

Generelt gælder der, at Simpsonsummen kan beregnes ud fra midtsummen og trapezsummen:

$$S(2\cdot n) = \frac{2}{3}\cdot M(n) + \frac{1}{3}\cdot T(n)$$

Tjek:

> $S(40) = \frac{2}{3}\cdot M(20) + \frac{1}{3}\cdot T(20)$

$$\frac{5164746205956800842117365 1}{2907213649433169510273840} = \frac{5164746205956800842117365 1}{2907213649433169510273840} \quad \textbf{(5.1.1)}$$

> $is(\textbf{(5.1.1)})$

$$true \quad \textbf{(5.1.2)}$$

Det stemmer altså, når $n = 20$ med den givne funktion $f(x)$.

> $S(2\cdot n) - \left(\frac{2}{3}\cdot M(n) + \frac{1}{3}\cdot T(n)\right)$

$$\frac{1}{n}\left(\frac{34}{9} + \frac{1}{6}\,\frac{1}{n^6}\left(32\,n^6\,(n+1) - 48\,n^5\,(n+1) + \frac{88}{3}\,n^4\,(n+1) + 24\,(n \right. \right. \quad \textbf{(5.1.3)}$$

$$+1)^2\,n^5 - 32\,n^4\,(n+1)^2 + \frac{32}{3}\,n^4\,(n+1)^3\Big) + \frac{2}{3}\,n\,\Psi\left(\frac{3}{2}\,n + \frac{1}{2}\right)$$

$$-\frac{1}{6}\,\frac{32\,n^6 - 24\,n^5 + 8\,n^4}{n^6} - \frac{2}{3}\,n\,\Psi\left(\frac{1}{2} + \frac{1}{2}\,n\right)$$

$$+\frac{1}{12}\,\frac{\frac{200}{3}\,n^7 - 40\,n^6 + \frac{16}{3}\,n^5}{n^6} + \frac{1}{3}\,n\,\Psi\left(\frac{3}{2}\,n\right) - \frac{1}{3}\,n\,\Psi\left(1 + \frac{1}{2}\,n\right)\Big)$$

$$-\frac{4}{3}\,\frac{\frac{1}{8}\,\frac{\frac{200}{3}\,n^7 - \frac{8}{3}\,n^5}{n^6} + \frac{1}{2}\,n\,\Psi\left(\frac{3}{2}\,n + \frac{1}{2}\right) - \frac{1}{2}\,n\,\Psi\left(\frac{1}{2} + \frac{1}{2}\,n\right)}{n}$$

$$-\frac{2}{3}\,\frac{1}{n}\left(\frac{17}{3} + \frac{1}{8}\,\frac{\frac{200}{3}\,n^7 - 40\,n^6 + \frac{16}{3}\,n^5}{n^6} + \frac{1}{2}\,n\,\Psi\left(\frac{3}{2}\,n\right)\right.$$

$$\left.\left. -\frac{1}{2}\,n\,\Psi\left(1 + \frac{1}{2}\,n\right)\right)\right)$$

> $simplify(\%)$

$$0 \quad \textbf{(5.1.4)}$$

Det stemmer altså for alle n med den givne funktion $f(x)$.

Læs nærmere om Ψ-funktionen (digamma), som anvender Γ-funktionen (gamma):

http://www.maplesoft.com/support/help/maple/view.aspx?path=Psi

http://www.maplesoft.com/support/help/maple/view.aspx?path=GAMMA

Formel for Simpsonsum udtrykt ved midtsum og trapezsum Generelt bevis med vilkårligt $f(x)$

Nu undersøges om formlen gælder for alle n for en vilkårlig funktion:

```
> unassign('f')
> S(2·n) - ( 2/3 ·M(n) + 1/3 ·T(n) )
```

$$\frac{\frac{1}{3}f(1)+\frac{4}{3}\sum_{i=1}^{n}f\left(1+\frac{2i-1}{n}\right)+\frac{2}{3}\sum_{i=1}^{n-1}f\left(1+\frac{2i}{n}\right)+\frac{1}{3}f(3)}{n} - \frac{4}{3}\,\frac{\sum_{i=0}^{n-1}f\left(1+\frac{2\left(i+\frac{1}{2}\right)}{n}\right)}{n} - \frac{2}{3}\,\frac{\frac{1}{2}f(1)+\sum_{i=1}^{n-1}f\left(1+\frac{2i}{n}\right)+\frac{1}{2}f(3)}{n} \qquad \textbf{(5.2.1)}$$

```
> simplify(%)
```

$$\frac{4}{3}\,\frac{\sum_{i=1}^{n}f\left(\frac{n+2i-1}{n}\right)-\left(\sum_{i=0}^{n-1}f\left(\frac{n+2i+1}{n}\right)\right)}{n} \qquad \textbf{(5.2.2)}$$

Her må man så lave lidt håndregning!

Man kan lave en substitution i den anden sum. Lad i være $j-1$:

$$\sum_{i=0}^{n-1}f\left(\frac{n+2\cdot i+1}{n}\right)=\sum_{j-1=0}^{n-1}f\left(\frac{n+2\cdot(j-1)+1}{n}\right)=\sum_{j=1}^{n}f\left(\frac{n+2\cdot(j-1)+1}{n}\right)=\sum_{j=1}^{n}f\left(\frac{n+2\cdot j-1}{n}\right)=\sum_{i=1}^{n}f\left(\frac{n+2\cdot i-1}{n}\right)$$

Dvs. første og anden sum i tælleren er identiske!

Derfor er formlen korrekt for alle n og for enhver funktion $f(x)$.

Formel for Simpsonsum udtrykt ved midtsum og trapezsum Bevis med $n = 1$

Undersøge om $S(2)=\frac{2}{3}\cdot M(1)+\frac{1}{3}\cdot T(1)$ for en vilkårlig funktion $f(x)$

Simpson

1) Løsning af 3 ligninger med 3 ubekendte

Simpsonsum skal lave en parabel gennem 3 punkter med $x: a, \frac{a+b}{2}, b$.
Derefter skal 2. gradspolynomiet integreres fra a til b:

```
> restart
> p1 := x→A·x^2 + B·x + C
```

$$p_1 := x \to A\,x^2 + B\,x + C \qquad \textbf{(5.3.1.1.1)}$$

```
> L := solve({p1(a) = f(a), p1((a+b)/2) = f((a+b)/2), p1(b) = f(b)}, {A, B, C})
```

$$L := \left\{ A = \frac{2\left(f(b) - 2f\left(\frac{1}{2}a + \frac{1}{2}b\right) + f(a)\right)}{a^2 - 2ab + b^2}, B = -\frac{1}{(-b+a)^2}\left(3f(b)\,a + f(b)\,b - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)a - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)b + f(a)\,a + 3f(a)\,b\right), C = \frac{1}{(-b+a)^2}\left(f(b)\,a^2 + f(b)\,ab - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)ab + f(a)\,ab + f(a)\,b^2\right)\right\} \qquad \textbf{(5.3.1.1.2)}$$

```
> A := rhs(L[1]); B := rhs(L[2]); C := rhs(L[3])
```

$$A := \frac{2\left(f(b) - 2f\left(\frac{1}{2}a + \frac{1}{2}b\right) + f(a)\right)}{a^2 - 2ab + b^2}$$

$$B := -\frac{1}{(-b+a)^2}\left(3f(b)\,a + f(b)\,b - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)a - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)b + f(a)\,a + 3f(a)\,b\right)$$

$$C := \frac{f(b)\,a^2 + f(b)\,ab - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)ab + f(a)\,ab + f(a)\,b^2}{(-b+a)^2} \qquad \textbf{(5.3.1.1.3)}$$

```
> p1(x)
```

$$\frac{2\left(f(b) - 2f\left(\frac{1}{2}a + \frac{1}{2}b\right) + f(a)\right)x^2}{a^2 - 2ab + b^2} - \frac{1}{(-b+a)^2}\left(\left(3f(b)\,a + f(b)\,b - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)a - 4f\left(\frac{1}{2}a + \frac{1}{2}b\right)b + f(a)\,a + 3f(a)\,b\right)x\right) \qquad \textbf{(5.3.1.1.4)}$$

$$+\frac{f(b)\,a^2+f(b)\,a\,b-4f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right)a\,b+f(a)\,a\,b+f(a)\,b^2}{(-b+a)^2}$$

> $S2_1 := \int_a^b p_1(x)\,dx$

$$S2_1 := \frac{2}{3}\,\frac{\left(f(b)-2f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right)+f(a)\right)(-a^3+b^3)}{a^2-2\,a\,b+b^2} \qquad \textbf{(5.3.1.1.5)}$$
$$-\frac{1}{2}\,\frac{1}{(-b+a)^2}\left(\left(3f(b)\,a+f(b)\,b-4f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right)a\right.\right.$$
$$\left.-4f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right)b+f(a)\,a+3f(a)\,b\right)(-a^2+b^2)\Big)$$
$$+\frac{1}{(-b+a)^2}\left(\left(f(b)\,a^2+f(b)\,a\,b-4f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right)a\,b\right.\right.$$
$$\left.\left.+f(a)\,a\,b+f(a)\,b^2\right)(-a+b)\right)$$

> $S2_1 := simplify(S2_1)$

$$S2_1 := -\frac{1}{6}\,f(b)\,a+\frac{1}{6}\,f(b)\,b-\frac{2}{3}\,f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right)a+\frac{2}{3}\,f\left(\frac{1}{2}\,a \qquad \textbf{(5.3.1.1.6)}$$
$$+\frac{1}{2}\,b\right)b-\frac{1}{6}\,f(a)\,a+\frac{1}{6}\,f(a)\,b$$

2) Løsning med polynomiel regression

> $with(Gym)$:

Lister med datapunkterne:
NB: ~ gør, at funktionen f virker på alle elementer i listen.

> $X := \left[a, \frac{a+b}{2}, b\right]$

$$X := \left[a, \frac{1}{2}\,a+\frac{1}{2}\,b, b\right] \qquad \textbf{(5.3.1.2.1)}$$

> $Y := f\sim(X)$

$$Y := \left[f(a), f\left(\frac{1}{2}\,a+\frac{1}{2}\,b\right), f(b)\right] \qquad \textbf{(5.3.1.2.2)}$$

> $PolyReg(X, Y, 2, x)$

Error, (in Statistics:-PolynomialFit) unable to store 'a' when datatype=float[8]

Det kan Maple åbenbart ikke løse, når a og b er ukendte parametre!

3) Løsning med formel for polynomium gennem givne punkter

Tak til Lars-Lund-Hansen, som er kommet med dette indspark.

Læs nærmere:

https://en.wikipedia.org/wiki/Polynomial_interpolation#Constructing_the_interpolation_polynomial

https://en.wikipedia.org/wiki/Lagrange_polynomial

$$p(x)=\frac{(x-x_1)(x-x_2)\cdots(x-x_n)}{(x_0-x_1)(x_0-x_2)\cdots(x_0-x_n)}y_0+\frac{(x-x_0)(x-x_2)\cdots(x-x_n)}{(x_1-x_0)(x_1-x_2)\cdots(x_1-x_n)}y_1+\ldots+\frac{(x-x_0)(x-x_1)\cdots(x-x_{n-1})}{(x_n-x_0)(x_n-x_1)\cdots(x_n-x_{n-1})}y_n$$
$$=\sum_{i=0}^{n}\left(\prod_{\substack{0\le j\le n\\ j\ne i}}\frac{x-x_j}{x_i-x_j}\right)y_i$$

> $N := 2$

$$N := 2 \qquad \textbf{(5.3.1.3.1)}$$

Interpolationsformlen fra WikiPedia implementeres således i Maple:
NB: listerne X og Y med datapunkterne har indeks fra 1 til 3, derfor må summationsvariablene i og j opdateres med 1.
piecewise gør, at når man kommer til at dividere med 0, så erstattes udtrykket med faktoren 1, som er uskyldig!

> $p_3 := unapply\left(\sum_{i=1}^{N+1}\left(\left(\prod_{j=1}^{N+1} piecewise\left(i=j,\,1,\,i\ne j,\,\frac{x-X[j]}{X[i]-X[j]}\right)\right)\cdot Y[i]\right),\,x\right):$

> $p_3(x)$

$$\frac{\left(x-\frac{1}{2}a-\frac{1}{2}b\right)(x-b)f(a)}{\left(\frac{1}{2}a-\frac{1}{2}b\right)(-b+a)}+\frac{(x-a)(x-b)f\left(\frac{1}{2}a+\frac{1}{2}b\right)}{\left(-\frac{1}{2}a+\frac{1}{2}b\right)\left(\frac{1}{2}a-\frac{1}{2}b\right)}+\frac{(x-a)\left(x-\frac{1}{2}a-\frac{1}{2}b\right)f(b)}{(-a+b)\left(-\frac{1}{2}a+\frac{1}{2}b\right)} \qquad \textbf{(5.3.1.3.2)}$$

> $S2_3 := \int_a^b p_3(x)\,dx$

$$S2_3 := \frac{1}{3}\left(\frac{f(a)}{\left(\frac{1}{2}a-\frac{1}{2}b\right)(-b+a)}+\frac{f\left(\frac{1}{2}a+\frac{1}{2}b\right)}{\left(-\frac{1}{2}a+\frac{1}{2}b\right)\left(\frac{1}{2}a-\frac{1}{2}b\right)}+\frac{f(b)}{(-a+b)\left(-\frac{1}{2}a+\frac{1}{2}b\right)}\right)(-a^3+b^3)$$
$$+\frac{1}{2}\left(\frac{\left(\frac{-\frac{1}{2}a-\frac{1}{2}b}{\frac{1}{2}a-\frac{1}{2}b}-\frac{b}{\frac{1}{2}a-\frac{1}{2}b}\right)f(a)}{-b+a}\right. \qquad \textbf{(5.3.1.3.3)}$$

$$+\frac{\left(-\frac{a}{-\frac{1}{2}a+\frac{1}{2}b}-\frac{b}{-\frac{1}{2}a+\frac{1}{2}b}\right)f\left(\frac{1}{2}a+\frac{1}{2}b\right)}{\frac{1}{2}a-\frac{1}{2}b}$$

$$+\frac{\left(-\frac{a}{-a+b}+\frac{-\frac{1}{2}a-\frac{1}{2}b}{-a+b}\right)f(b)}{-\frac{1}{2}a+\frac{1}{2}b}\Bigg)\left(-a^2+b^2\right)$$

$$-\frac{\left(-\frac{1}{2}a-\frac{1}{2}b\right)bf(a)\,(-a+b)}{\left(\frac{1}{2}a-\frac{1}{2}b\right)(-b+a)}$$

$$+\frac{abf\left(\frac{1}{2}a+\frac{1}{2}b\right)(-a+b)}{\left(-\frac{1}{2}a+\frac{1}{2}b\right)\left(\frac{1}{2}a-\frac{1}{2}b\right)}-\frac{a\left(-\frac{1}{2}a-\frac{1}{2}b\right)f(b)}{-\frac{1}{2}a+\frac{1}{2}b}$$

> $S2_3 := simplify(S2_3)$

$$S2_3 := -\frac{1}{6}f(b)\,a+\frac{1}{6}f(b)\,b-\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)a+\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)b-\frac{1}{6}f(a)\,a+\frac{1}{6}f(a)\,b \quad \textbf{(5.3.1.3.4)}$$

Er $S2_1 = S2_3$?

> $S2_1$

$$-\frac{1}{6}f(b)\,a+\frac{1}{6}f(b)\,b-\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)a+\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)b-\frac{1}{6}f(a)\,a+\frac{1}{6}f(a)\,b \quad \textbf{(5.3.1.4.1)}$$

> $S2_3$

$$-\frac{1}{6}f(b)\,a+\frac{1}{6}f(b)\,b-\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)a+\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)b-\frac{1}{6}f(a)\,a+\frac{1}{6}f(a)\,b \quad \textbf{(5.3.1.4.2)}$$

> $S2_1 - S2_3$

$$0 \quad \textbf{(5.3.1.4.3)}$$

Ja begge metoder giver samme Simpsonsum!

Midtsum

Midtsummen $M(1)$ beregnes ud areal af et rektangel:

> $M1 := f\left(\frac{a+b}{2}\right)\cdot(b-a)$

$$M1 := f\left(\frac{1}{2}a+\frac{1}{2}b\right)(-a+b) \quad \textbf{(5.3.2.1)}$$

Trapezsum

Trapezsummen $T(1)$ beregnes ud fra areal af et trapez:

> $T1 := \frac{f(a)+f(b)}{2}\cdot(b-a)$

$$T1 := \frac{1}{2}\,(f(a)+f(b))\,(-a+b) \qquad \textbf{(5.3.3.1)}$$

Stemmer formlen $S2 = \frac{2}{3}\cdot M1 + \frac{1}{3}\cdot T1$ *?*

Sammenligning af formlerne:

> $S2_1$

$$-\frac{1}{6}f(b)\,a+\frac{1}{6}f(b)\,b-\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)a+\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)b-\frac{1}{6}f(a)\,a+\frac{1}{6}f(a)\,b \qquad \textbf{(5.3.4.1)}$$

> $\frac{2}{3}\cdot M1+\frac{1}{3}\cdot T1$

$$\frac{2}{3}f\left(\frac{1}{2}a+\frac{1}{2}b\right)(-a+b)+\frac{1}{6}\,(f(a)+f(b))\,(-a+b) \qquad \textbf{(5.3.4.2)}$$

> $simplify\left(S2_1-\left(\frac{2}{3}\cdot M1+\frac{1}{3}\cdot T1\right)\right)$

$$0 \qquad \textbf{(5.3.4.3)}$$

Ja, formlen stemmer!